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New Semiclassical Nonabelian Vertex Operators for Chiral and Nonchiral WZW Theory

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ABSTRACT

We supplement the discussion of Moore and Reshetikhin and others by finding new semiclassical nonabelian vertex operators for the chiral, antichiral and nonchiral primary fields of WZW theory. These new nonabelian vertex operators are the natural generalization of the familiar abelian vertex operators: They involve only the representation matrices of Lie g, the currents of affine $(g \times g)$ and certain chiral and antichiral zero modes, and they reduce to the abelian vertex operators in the limit of abelian algebras. Using the new constructions, we also discuss semiclassical operator product expansions, braid relations and relations to the known form of the semiclassical affine-Sugawara conformal blocks.

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1 Introduction

Affine Lie algebra [1,2] is the basis of a very large set of conformal field theories called the affine-Virasoro constructions [3,4] which include the affine-Sugawara constructions [2,5,6,7], the coset constructions [2,5,8] and the irrational conformal field theories [3,9,10]. Among these, the simplest theories are the affine-Sugawara constructions and their corresponding WZW actions [11,12], which have often served as a testing ground for new ideas in conformal field theory. See Ref.[10] for a more detailed history of affine Lie algebra and the affine-Virasoro constructions.

Vertex operator constructions (see for example [13-22]) are explicit realizations (using the familiar abelian vertex operators [23]) of the fermions, currents and primary fields of affine Lie algebras and conformal field theories. The first vertex operator constructions [13-15] were the vertex operator constructions of world-sheet fermions and level one of untwisted $SU(n)$, which was also the first construction of current-algebraic internal symmetry from compactified dimensions on the string. The generalization [17,18] of this construction to level one of simply laced g plays a central role in the formulation of the heterotic string [24]. More generally, the vertex operator constructions may be divided into the explicitly unitary constructions [13-18] and the constructions of (bosonized) Wakimoto type [19-22], which must be projected onto unitary subspaces.

In this paper, we will supplement the discussion of Moore and Reshetikin [25] and others [26-41] by finding a new explicit semiclassical (high-level) realization of the chiral, antichiral and nonchiral primary fields of WZW theory. The realization is obtained by semiclassical solution of known [26,25] operator differential equations for the chiral primary fields (so that no unitary projection is needed), and the results are recognized as the semiclassical form of new nonabelian vertex operators which are the natural generalization of the familiar abelian vertex operator: In particular, the new nonabelian vertex operators involve only the representation matrices of Lie g, the currents of affine $(g\times g)$ and certain chiral and antichiral zero modes, and they reduce to the abelian vertex operators in the limit of abelian algebras.

A central feature of the construction is the identification of the chiral and antichiral zero modes which, thru the semiclassical order we have studied, are seen to carry the full action of the quantum group. We are also able to identify the classical limit of the nonchiral product of the zero modes as the classical group element.

As applications of our construction, we compute the semiclassical OPE's of all the primary fields and compare the averages of the primary fields to the known [42] forms of the semiclassical affine-Sugawara conformal blocks and WZW correlators. The relation of the construction to semiclassical crossing and braiding is also discussed.

2 Affine Lie Algebra and WZW Theory

In this section we review some basic facts about affine Lie algebra [1,2] and the affine-Sugawara constructions [2,5,6,7] which provide the algebraic description of WZW theory [12].

We begin with the algebra of affine $(g \times g)$, which consists of two commuting copies of affine g,

$$[J_a(m), J_b(n)] = if_{ab}{}^c J_c(m+n) + km\eta_{ab}\delta_{m+n,0} \tag{2.1a}$$

$$[\bar{J}_a(m), \bar{J}_b(n)] = if_{ab}{}^c \bar{J}_c(m+n) + km\eta_{ab}\delta_{m+n,0} \tag{2.1b}$$

$$[J_a(m), \bar{J}_b(n)] = 0 \quad , \quad m,n \in \mathbb{Z} \quad , \quad a,b,c = 1 \ldots \dim g \tag{2.1c}$$

where $f_{ab}{}^c$ and η_{ab} are the structure constants and Killing metric of g and k is the level of the affine algebra. For simplicity we generally assume here that g is compact, though most of the statements below apply as well to the noncompact extensions of g. The affine vacuum state $|0\rangle$ satisfies

$$J_a(m \geq 0)|0\rangle = \bar{J}_a(m \geq 0)|0\rangle = 0 \quad . \tag{2.2}$$

In terms of these current modes, the local chiral and antichiral currents are defined as

$$J_a(z) \equiv \sum_{m\in\mathbb{Z}} J_a(m) z^{-m-1} \quad , \quad \bar{J}_a(\bar{z}) \equiv \sum_{m\in\mathbb{Z}} \bar{J}_a(m) \bar{z}^{-m-1} \tag{2.3}$$

where z is the complex Euclidian world-sheet coordinate and $\bar{z}$ is the complex conjugate of z.

We will also need the primary fields $g(\mathcal{T}, \bar{z}, z)$ of affine $(g \times g)$, which transform under the current modes as

$$[J_a(m), g(\mathcal{T}, \bar{z}, z)_\alpha{}^\beta] = g(\mathcal{T}, \bar{z}, z)_\alpha{}^\gamma z^m (\mathcal{T}_a)_\gamma{}^\beta \tag{2.4a}$$

$$[\bar{J}_a(m), g(\mathcal{T}, \bar{z}, z)_\alpha{}^\beta] = -\bar{z}^m (\mathcal{T}_a)_\alpha{}^\gamma g(\mathcal{T}, \bar{z}, z)_\gamma{}^\beta \tag{2.4b}$$

$$[\mathcal{T}_a, \mathcal{T}_b] = if_{ab}{}^c \mathcal{T}_c \quad , \quad \alpha, \beta = 1 \ldots \dim \mathcal{T} \tag{2.4c}$$

where $\mathcal{T}$ is a matrix irrep of g. The primary fields $g(\mathcal{T}, \bar{z}, z)$ and the currents $J, \bar{J}$ may be understood [43] respectively as the (reduced) affine Lie group element and the (reduced) left- and right-invariant affine Lie derivatives on the manifold of the affine Lie group. Acting on the affine vacuum, the primary fields create the primary states $\psi(\mathcal{T})$ of affine $(g \times g)$,

$$\psi_\alpha{}^\beta(\mathcal{T}) \equiv g(\mathcal{T}, 0, 0)_\alpha{}^\beta |0\rangle \tag{2.5a}$$

$$J_a(m \geq 0)\psi_\alpha{}^\beta(\mathcal{T}) = \delta_{m,0} \psi_\alpha{}^\gamma(\mathcal{T})(\mathcal{T}_a)_\gamma{}^\beta \tag{2.5b}$$

$$\bar{J}_a(m \geq 0)\psi_\alpha{}^\beta(\mathcal{T}) = -\delta_{m,0} (\mathcal{T}_a)_\alpha{}^\gamma \psi_\gamma{}^\beta(\mathcal{T}) \tag{2.5c}$$

which transform in irrep $\mathcal{T} \otimes \bar{\mathcal{T}}$ of $(g \times g)$. A coordinate-space representation of these states is given in Ref.[43].

The stress tensor of WZW theory is composed of the chiral and antichiral affine-Sugawara constructions

$$T_g(z) = L_g^{ab} \,{}^*_*J_a(z)J_b(z)\,{}^*_* = \sum_{m\in\mathbb{Z}} L_g(m)z^{-m-2} \tag{2.6a}$$

$$\bar{T}_g(\bar{z}) = L_g^{ab} \,{}^*_*\bar{J}_a(\bar{z})\bar{J}_b(\bar{z})\,{}^*_* = \sum_{m\in\mathbb{Z}} \bar{L}_g(m)\bar{z}^{-m-2} \tag{2.6b}$$

$$L_g^{ab} = \frac{\eta^{ab}}{2k+Q_g} \tag{2.6c}$$

$$c_g = \bar{c}_g = \frac{2k\dim g}{2k+Q_g} \tag{2.6d}$$

whose modes $L_g(m)$ and $\bar{L}_g(m)$ satisfy two commuting Virasoro algebras with central charges c_g and $\bar{c}_g$. Here, Q_g is the quadratic Casimir of the adjoint and L_g^{ab} is called the inverse inertia tensor of the affine-Sugawara construction. The currents $J, \bar{J}$ and the affine-primary fields $g(\mathcal{T},\bar{z},z)$ are also Virasoro primary fields under $(T_g,\bar{T}_g)$ with conformal weights $(1,0)$, $(0,1)$ and $(\Delta^g(\mathcal{T}),\Delta^g(\mathcal{T}))$ respectively. The affine-Sugawara conformal weight $\Delta^g(\mathcal{T})$ is given by

$$L_g^{ab}\mathcal{T}_a\mathcal{T}_b = \Delta^g(\mathcal{T})\mathbb{1} \quad , \quad \Delta^g(\mathcal{T}) = \frac{Q(\mathcal{T})}{2k+Q_g} \tag{2.7}$$

with $Q(\mathcal{T})$ the quadratic Casimir of $\mathcal{T}$. In what follows, the affine- and Virasoro-primary fields $g(\mathcal{T},\bar{z},z)$ are generally called the WZW primary fields.

In the WZW action, the classical analogue of the WZW primary field $g(\mathcal{T},\bar{z},z)$ appears as the unitary Lie group element in irrep $\mathcal{T}$ of g. We shall see below however that the WZW primary field $g(\mathcal{T},\bar{z},z)$ is a unitary operator only in the extreme semiclassical limit, due to normal ordering in the quantum theory.

Differential equations

Because they are also primary under $(T_g,\bar{T}_g)$, the WZW primary fields $g(\mathcal{T},\bar{z},z)$ satisfy the operator relations

$$\partial g(\mathcal{T},\bar{z},z) = [L_g(-1), g(\mathcal{T},\bar{z},z)] \quad , \quad \bar{\partial} g(\mathcal{T},\bar{z},z) = [\bar{L}_g(-1), g(\mathcal{T},\bar{z},z)] \tag{2.8a}$$

$$L_g(-1) = 2L_g^{ab}\sum_{m\geq 0} J_a(-m-1)J_b(m) \quad , \quad \bar{L}_g(-1) = 2L_g^{ab}\sum_{m\geq 0} \bar{J}_a(-m-1)\bar{J}_b(m) \tag{2.8b}$$

and, using (2.8b) in (2.8a), one finds the partial differential equations (PDE's) for the WZW primary fields [7]

$$\partial g(\mathcal{T},\bar{z},z)_\alpha{}^\beta = 2L_g^{ab} : g(\mathcal{T},\bar{z},z)_\alpha{}^\gamma J_a(z) : (\mathcal{T}_b)_\gamma{}^\beta \tag{2.9a}$$

$$\bar{\partial} g(\mathcal{T},\bar{z},z)_\alpha{}^\beta = -2L_g^{ab}(\mathcal{T}_b)_\alpha{}^\gamma : g(\mathcal{T},\bar{z},z)_\gamma{}^\beta \bar{J}_a(\bar{z}) : \quad . \tag{2.9b}$$

In verifying (2.9), one finds that the normal-ordering prescription is

$$: g(\mathcal{T},\bar{z},z)J_a(z) : = J_a^-(z)g(\mathcal{T},\bar{z},z) + g(\mathcal{T},\bar{z},z)(J_a^+(z) + J_a(0)\frac{1}{z}) \tag{2.10a}$$

$$: g(\mathcal{T},\bar{z},z)\bar{J}_a(\bar{z}) : = \bar{J}_a^-(\bar{z})g(\mathcal{T},\bar{z},z) + g(\mathcal{T},\bar{z},z)(\bar{J}_a^+(\bar{z}) + \bar{J}_a(0)\frac{1}{\bar{z}}) \tag{2.10b}$$

$$J_a^\pm(z) \equiv \sum_{m>0} J_a(\pm m) z^{\mp m-1} \quad , \quad \bar{J}_a^\pm(\bar{z}) \equiv \sum_{m>0} \bar{J}_a(\pm m)\bar{z}^{\mp m-1} \tag{2.10c}$$

where the positive and negative modes of the currents are collected in the definitions (2.10c) and $J_a(0)$, $\bar{J}_a(0)$ are the zero modes. It is easily checked that the PDE's (2.9a) and (2.9b) are consistent (that is, (2.9) defines a flat connection) because J commutes with $\bar{J}$. It will also be convenient to define the integrated quantities

$$Q_a^\pm(z) \equiv \pm i \sum_{m>0} \frac{J_a(\pm m)}{m} z^{\mp m} \quad , \quad \bar{Q}_a^\pm(\bar{z}) \equiv \pm i \sum_{m>0} \frac{\bar{J}_a(\pm m)}{m} \bar{z}^{\mp m} \tag{2.11}$$

for use below.

Ultimately, one is interested in the n-point WZW correlators of the WZW primary fields

$$A_g(\mathcal{T},\bar{z},z) = \langle 0|g(\mathcal{T}^1,\bar{z}_1,z_1)\cdots g(\mathcal{T}^n,\bar{z}_n,z_n)|0\rangle \tag{2.12}$$

which satisfy the $(g\times g)$-global Ward identities

$$\sum_{i=1}^n \mathcal{T}_a^i A_g(\mathcal{T},\bar{z},z) = A_g(\mathcal{T},\bar{z},z)\sum_{i=1}^n \mathcal{T}_a^i = 0 \quad . \tag{2.13}$$

The WZW correlators (2.12) also satisfy $SL(2,\mathrm{I\!R})\times SL(2,\mathrm{I\!R})$ Ward identities and the chiral and antichiral Knizhnik-Zamolodchikov (KZ) equations [7]

$$\partial_i A_g(\mathcal{T},\bar{z},z) = A_g(\mathcal{T},\bar{z},z) 2L_g^{ab}\sum_{j\neq i}\frac{\mathcal{T}_a^i\mathcal{T}_b^j}{z_{ij}} \tag{2.14a}$$

$$\bar{\partial}_i A_g(\mathcal{T},\bar{z},z) = 2L_g^{ab}\sum_{j\neq i}\frac{\mathcal{T}_a^i\mathcal{T}_b^j}{\bar{z}_{ij}} A_g(\mathcal{T},\bar{z},z) \tag{2.14b}$$

which follow from the PDE's (2.9)

Semiclassical expansion

In this paper, we will be interested primarily in the semiclassical or high-level expansion [44,45,10,43,42] of the low-spin sector of the theory, which is defined by the level-orders:

$$J_a(0) = \mathcal{O}(k^0) \quad , \quad J_a(m\neq 0) = \mathcal{O}(k^{1/2}) \tag{2.15a}$$

$$\bar{J}_a(0) = \mathcal{O}(k^0) \quad , \quad \bar{J}_a(m\neq 0) = \mathcal{O}(k^{1/2}) \tag{2.15b}$$

$$\mathcal{T}_a = \mathcal{O}(k^0) \tag{2.15c}$$

$$L_g^{ab} = \frac{\eta^{ab}}{2k} + \mathcal{O}(k^{-2}) = \mathcal{O}(k^{-1}) \quad , \quad \Delta_g(\mathcal{T}) = \frac{Q(\mathcal{T})}{2k} + \mathcal{O}(k^{-2}) = \mathcal{O}(k^{-1}) \quad . \tag{2.15d}$$

In this case, the corresponding semiclassical affine-Sugawara conformal blocks and WZW correlators have been worked out in Ref.[42]. In particular, the solution for the high-level n-point WZW correlators (2.12) is

$$A_g(\mathcal{T}, \bar{z}, z) = \left(1\!\!1 + 2L_g^{ab} \sum_{i<j}^{n} \mathcal{T}_a^i \mathcal{T}_b^j \ln \bar{z}_{ij} \right) I_g^n \left(1\!\!1 + 2L_g^{ab} \sum_{i<j}^{n} \mathcal{T}_a^i \mathcal{T}_b^j \ln z_{ij} \right) + \mathcal{O}(k^{-2}) \tag{2.16}$$

$$= \left(1\!\!1 + 4L_g^{ab} \sum_{i<j}^{n} \mathcal{T}_a^i \mathcal{T}_b^j \ln |z_{ij}| \right) I_g^n + \mathcal{O}(k^{-2})$$

where I_g^n is the n-point Haar integral

$$(I_g^n)_\alpha{}^\beta = \int \mathrm{d}\mathcal{G}\, \mathcal{G}(\mathcal{T}^1)_{\alpha_1}{}^{\beta_1} \cdots \mathcal{G}(\mathcal{T}^n)_{\alpha_n}{}^{\beta_n} \tag{2.17}$$

over unitary Lie group elements $\mathcal{G}(\mathcal{T})$ in matrix irrep $\mathcal{T}$ of g. The Haar integral is invariant under $g \times g$ and satisfies $(I_g^n)^2 = I_g^n$, so that I_g^n is the projector onto the g-invariant subspace of $\mathcal{T}^1 \otimes \cdots \otimes \mathcal{T}^n$.

Factorization of the WZW primary fields

Our goal in this paper is to solve the algebra (2.4) and the PDE (2.9) to obtain the explicit semiclassical form (the WZW vertex operators) of the WZW primary fields $g(\mathcal{T}, \bar{z}, z)$. To this end, we look for solutions in the factorized form

$$g(\mathcal{T}, \bar{z}, z)_\alpha{}^\beta = g_-(\mathcal{T}, \bar{z})_\alpha{}^A g_+(\mathcal{T}, z)_A{}^\beta \tag{2.18}$$

where $A, B = 1 \ldots \dim \mathcal{T}$ are the quantum group indices discussed by Moore and Reshetikhin [25] and others [26-41]. In what follows, g_+ and g_- will be referred to as the chiral and antichiral primary fields and/or the chiral and antichiral vertex operators of the theory.

To find such factorized solutions, we assume that g_+ and g_- are affine-primary fields under J and $\bar{J}$ respectively,

$$[J_a(m), g_+(\mathcal{T}, z)] = g_+(\mathcal{T}, z) z^m \mathcal{T}_a \tag{2.19a}$$

$$[\bar{J}_a(m), g_-(\mathcal{T}, \bar{z})] = -\bar{z}^m \mathcal{T}_a\, g_-(\mathcal{T}, \bar{z}) \tag{2.19b}$$

which solves (2.4a,b) so long as

$$[J_a(m), g_-(\mathcal{T}, \bar{z})] g_+(\mathcal{T}, z) = g_-(\mathcal{T}, \bar{z}) [\bar{J}_a(m), g_+(\mathcal{T}, z)] = 0 \quad . \tag{2.20}$$

Then the PDE's (2.9a,b) are solved by the ordinary differential equations (ODE's) for the chiral and antichiral primary fields [26,25]

$$\partial g_+(\mathcal{T}, z) = 2L_g^{ab} : g_+(\mathcal{T}, z) J_a(z) : \mathcal{T}_b \tag{2.21a}$$

$$\bar{\partial}g_-(\mathcal{T},\bar{z}) = -2L_g^{ab}\mathcal{T}_b : g_-(\mathcal{T},\bar{z})\bar{J}_a(\bar{z}) : \tag{2.21b}$$

where normal ordering is defined by (2.10) with $g \to g_\pm$.

The explicit semiclassical forms of $g_\pm$ (the chiral and antichiral vertex operators) obtained below by solving the ODE's (2.21) will reproduce the semiclassical WZW correlators (2.12) in the factorized form

$$A_g(\mathcal{T},\bar{z},z)_\alpha{}^\beta = A_g^-(\mathcal{T},\bar{z})_\alpha{}^A A_g^+(\mathcal{T},z)_A{}^\beta \tag{2.22a}$$

$$A_g^+(\mathcal{T},z)_A{}^\beta = {}_+\langle 0|g_+(\mathcal{T}^1,z_1)_{A_1}{}^{\beta_1}\cdots g_+(\mathcal{T}^n,z_n)_{A_n}{}^{\beta_n}|0\rangle_+ \tag{2.22b}$$

$$A_g^-(\mathcal{T},\bar{z})_\alpha{}^A = {}_-\langle 0|g_-(\mathcal{T}^1,\bar{z}_1)_{\alpha_1}{}^{A_1}\cdots g_-(\mathcal{T}^n,\bar{z}_n)_{\alpha_n}{}^{A_n}|0\rangle_- \tag{2.22c}$$

$$\sum_{i=1}^n \mathcal{T}_a^i A_g^-(\mathcal{T},\bar{z}) = A_g^+(\mathcal{T},z)\sum_{i=1}^n \mathcal{T}_a^i = 0 \tag{2.22d}$$

where A_g^+ and A_g^- are the chiral and antichiral correlators. Here we have also assumed factorization of the vacuum state

$$|0\rangle = |0\rangle_-|0\rangle_+ \tag{2.23}$$

into the affine vacua $|0\rangle_+$ and $|0\rangle_-$ of J and $\bar{J}$. We will generally suppress the subscripts on the vacua, which will be clear in context.

The problem

The ODE's (2.21) for the primary fields $g_\pm$ can be solved by iteration of equivalent integral equations, e.g.

$$g_+(\mathcal{T},z) = g_+(\mathcal{T},z_0) + \int_{z_0}^{z} \mathrm{d}z'\, 2L_g^{ab} : g_+(\mathcal{T},z')J_a(z') : \mathcal{T}_b \tag{2.24}$$

where z_0 is a regular reference point. As noted by Moore and Reshetikhin [25], the iterative solution of (2.24) is not directly useful because the leading term $g_+(\mathcal{T},z) \simeq g_+(\mathcal{T},z_0)$ of this expansion would give singular chiral correlators $\langle g_+(\mathcal{T}^1,z_0)\cdots g_+(\mathcal{T}^n,z_0)\rangle = \infty$. In fact, the iterative solution is somewhat misleading because differentiation of (2.24) by z_0 shows that $g_+(\mathcal{T},z)$ is independent of z_0 when the initial condition $g_+(\mathcal{T},z_0)$ itself satisfies the original equation

$$\partial_{z_0} g_+(\mathcal{T},z_0) = 2L_g^{ab} : g_+(\mathcal{T},z_0)J_a(z_0) : \mathcal{T}_b \quad . \tag{2.25}$$

Using this fact, we shall see below that the iterative solutions can be rearranged into well-defined z_0 and $\bar{z}_0$-independent semiclassical expansions of the primary fields $g_\pm$.

3 Abelian Vertex Operators

Because it is the simplest, we consider first the case of decompactified abelian $g = U(1)^N$, for which the chiral system takes the form

$$g_+(\mathcal{T}, z) = g_+(\mathcal{T}, z_0) + \int_{z_0}^{z} \mathrm{d}z' \; 2L_g^{ab} : g_+(\mathcal{T}, z')J_a(z') : \mathcal{T}_b \tag{3.1a}$$

$$[J_a(m), J_b(n)] = km\eta_{ab}\delta_{m+n,0} \tag{3.1b}$$

$$[J_a(m), g_+(\mathcal{T}, z)] = g_+(\mathcal{T}, z)z^m\mathcal{T}_a \tag{3.1c}$$

$$[\mathcal{T}_a, \mathcal{T}_b] = 0 \quad , \quad a, b = 1 \ldots N \tag{3.1d}$$

$$L_g^{ab} = \frac{\eta^{ab}}{2k} \quad , \quad \Delta_g(\mathcal{T}) = \frac{\eta^{ab}\mathcal{T}_a\mathcal{T}_b}{2k} \quad . \tag{3.1e}$$

Here, the representation matrices $\mathcal{T}_a$ (the momenta), and hence g_+, are 1×1 (i.e. numbers), and we shall see that the solution $g_+(\mathcal{T}, z)$ of the system (3.1) can be rearranged into the familiar z_0-independent abelian vertex operator of the open bosonic string. Essentially the same vertex operators (with diagonal matrix $\mathcal{T}_a$'s) are obtained for compactified abelian algebras such as the Cartan subalgebra of a Lie algebra or other momentum lattices.

The solution of the integral equation (3.1a) is obtained on inspection as

$$\begin{aligned} g_+(\mathcal{T}, z) = &\exp\left(\int_{z_0}^{z} \mathrm{d}z' \, 2L_g^{ab}\mathcal{T}_a J_b^-(z')\right) g_+(\mathcal{T}, z_0) \\ &\times \exp\left(\int_{z_0}^{z} \mathrm{d}z' \, 2L_g^{ab}\mathcal{T}_a J_b(0)/z'\right) \exp\left(\int_{z_0}^{z} \mathrm{d}z' \, 2L_g^{ab}\mathcal{T}_a J_b^+(z')\right) \end{aligned} \tag{3.2}$$

where z_0 is the reference point and $J^\pm(z)$ are defined in (2.10c). Performing the integrations in (3.2), the result may be rearranged in the z_0-independent form

$$g_+(\mathcal{T}, z) = V_-(\mathcal{T}, z)G_+(\mathcal{T})V_0(\mathcal{T}, z)V_+(\mathcal{T}, z) \tag{3.3a}$$

$$V_\pm(\mathcal{T}, z) \equiv \exp(2iL_g^{ab}\mathcal{T}_a Q_b^\pm(z)) \tag{3.3b}$$

$$V_0(\mathcal{T}, z) \equiv \exp(2L_g^{ab}\mathcal{T}_a J_b(0)\ln z) = z^{2L_g^{ab}\mathcal{T}_a J_b(0)} \tag{3.3c}$$

where $Q^\pm$ is defined in (2.11). In this form, we have collected all z_0-dependent factors from the integrations into the constant quantity

$$G_+(\mathcal{T}) \equiv V_-(\mathcal{T}, z_0)^{-1}g_+(\mathcal{T}, z_0)V_0(\mathcal{T}, z_0)^{-1}V_+(\mathcal{T}, z_0)^{-1} \tag{3.4a}$$

$$\partial_z G_+ = \partial_{z_0} G_+ = 0 \tag{3.4b}$$

which is in fact independent of z_0 because $\partial_{z_0} g_+(\mathcal{T}, z_0) = 2L_g^{ab} : g_+(\mathcal{T}, z_0)J_a(z_0) : \mathcal{T}_b$. In what follows, we refer to the quantity $G_+(\mathcal{T})$ as the chiral zero mode of the chiral vertex operator.

The z_0-independent solution (3.3a) has the form of the usual abelian vertex operator, but we do not yet know the algebra of the zero mode $G_+(\mathcal{T})$ with the currents.

In fact, this algebra is determined by the system. To see this, invert (3.3a) to write the zero mode in terms of the primary field

$$G_+(\mathcal{T}) = V_-(\mathcal{T},z)^{-1} g_+(\mathcal{T},z) V_+(\mathcal{T},z)^{-1} V_0(\mathcal{T},z)^{-1} \quad . \tag{3.5}$$

Then, the algebra of the currents with the zero mode

$$[J_a(m), G_+(\mathcal{T})] = G_+(\mathcal{T})\mathcal{T}_a \delta_{m,0} \tag{3.6}$$

is obtained straightforwardly from (3.5), (3.1c) and the current algebra (3.1b).

The algebra (3.6) is solved by

$$G_+(\mathcal{T}) = \mathrm{e}^{iq^a \mathcal{T}_a} \quad , \quad [q^a, J_b(m)] = i\delta_b^a \delta_{m,0} \tag{3.7}$$

so that the solution (3.3a) may be written as

$$g_+(\mathcal{T},z) = \mathrm{e}^{2iL_g^{ab}\mathcal{T}_a q_b} z^{2L_g^{ab}\mathcal{T}_a J_b(0)} \exp(2iL_g^{ab}\mathcal{T}_a Q_b^-(z)) \exp(2iL_g^{ab}\mathcal{T}_a Q_b^+(z)) \tag{3.8}$$

where $q^a = 2L_g^{ab} q_b$. With the conventional identification of the metric G_{ab} and its inverse G^{ab},

$$[J_a(m), J_b(n)] = mG_{ab}\delta_{m+n,0} \tag{3.9a}$$

$$G_{ab} = k\eta_{ab} \quad , \quad G^{ab} = \frac{\eta^{ab}}{k} = 2L_g^{ab} \tag{3.9b}$$

the result (3.8) is recognized as the familiar abelian vertex operator with momentum $\mathcal{T}_a$.

As an introduction to the non-abelian case below, we list some well-known properties of the abelian vertex operators.

A. Affine-primary states. On the affine vacuum, the vertex operators create the affine-primary states

$$|\mathcal{T}\rangle \equiv g_+(\mathcal{T},0)|0\rangle = G_+(\mathcal{T})|0\rangle = \mathrm{e}^{iq^a\mathcal{T}_a}|0\rangle \tag{3.10a}$$

$$J_a(m \geq 0)|\mathcal{T}\rangle = \delta_{m,0}|\mathcal{T}\rangle\mathcal{T}_a \tag{3.10b}$$

which are nothing but the chiral zero modes $G_+(\mathcal{T})$ on the vacuum.

B. Intrinsic monodromy. When z is taken around a closed loop, one finds the intrinsic monodromy relation

$$g_+(\mathcal{T}, ze^{2\pi i}) = g_+(\mathcal{T},z) e^{4\pi i L_g^{ab}\mathcal{T}_a J_b(0)} \quad . \tag{3.11}$$

Using the algebra (3.1c) and the g-global invariance (momentum conservation) in (2.13), the operator relation (3.11) implies the correlator monodromies

$$\begin{aligned} &\langle 0|g_+(\mathcal{T}^1,z_1)\cdots g_+(\mathcal{T}^i, z_i e^{2\pi i})\cdots g_+(\mathcal{T}^n,z_n)|0\rangle \\ &\quad = \langle 0|g_+(\mathcal{T}^1,z_1)\cdots g_+(\mathcal{T}^i,z_i)\cdots g_+(\mathcal{T}^n,z_n)|0\rangle \mathrm{e}^{-4\pi i L_g^{ab}\mathcal{T}_a^i \sum_{j\leq i}^n \mathcal{T}_b^j} \end{aligned} \tag{3.12}$$

and it is not difficult to see that these phases are trivial for the open bosonic string.

C. Operator products and expansions. The operator product of two chiral vertex operators can be written as

$$g_+(\mathcal{T}^1,z_1)g_+(\mathcal{T}^2,z_2) = :g_+(\mathcal{T}^1,z_1)g_+(\mathcal{T}^2,z_2): z_{12}^{2L_g^{ab}\mathcal{T}_a^1\mathcal{T}_b^2} \tag{3.13a}$$

$$\begin{aligned} :g_+(\mathcal{T}^1,z_1)g_+(\mathcal{T}^2,z_2): &\equiv G_+(\mathcal{T}^1)G_+(\mathcal{T}^2)V_0(\mathcal{T}^1,z_1)V_0(\mathcal{T}^2,z_2) \\ &\times V_-(\mathcal{T}^1,z_1)V_-(\mathcal{T}^2,z_2)V_+(\mathcal{T}^1,z_1)V_+(\mathcal{T}^2,z_2) \end{aligned} \tag{3.13b}$$

where $z_{12}=z_1-z_2$ and the normal-ordered product in (3.13b) puts the zero modes G_+ to the left. The closed algebra of the zero modes

$$G_+(\mathcal{T}^1)G_+(\mathcal{T}^2) = G_+(\mathcal{T}^3) \quad , \quad \mathcal{T}^3 \equiv \mathcal{T}^1+\mathcal{T}^2 \tag{3.14}$$

then implies the OPE of two vertex operators

$$\begin{aligned} g_+(\mathcal{T}^1,z_1)g_+(\mathcal{T}^2,z_2) = z_{12}^{\Delta^g(\mathcal{T}^3)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)} \Big\{ & g_+(\mathcal{T}^3,z_2) \\ & + \sum_{r=0}^{\infty} \frac{z_{12}^{r+1}}{(r+1)!} 2L_g^{ab} :g_+(\mathcal{T}^3,z_2)\partial_2^r J_b(z_2): \mathcal{T}_a^1 \\ & + \sum_{r,s=0}^{\infty} \frac{z_{12}^{r+s+2}}{(r+s+2)!} 4L_g^{ab}L_g^{cd} :g_+(\mathcal{T}^3,z_2)\partial_2^r[\partial_2^s J_b(z_2)J_c(z_2)]: \mathcal{T}_d^1\mathcal{T}_a^1 \\ & + \text{higher affine secondaries} \Big\} \end{aligned} \tag{3.15}$$

where we have used the expression (3.1e) for the affine-Sugawara conformal weights. In (3.15), the normal-ordered product $:g_+J:$ is the chiral analogue of (2.10a), while

$$:g_+(\mathcal{T},z)J_a(z)J_b(z): = :(:g_+(\mathcal{T},z)J_b(z):)J_a(z): \tag{3.16}$$

is defined iteratively from (2.10a). The term "higher affine secondaries" stands for the fields $:g_+J^p:$, $p\geq 3$ and derivatives thereof.

The OPE (3.15) follows directly from (3.13), without using the ODE (2.21a). Using the ODE however, one may rearrange (3.15) in terms of the affine-primary fields (and their Virasoro descendants) plus those affine-secondary fields which are Virasoro primary (and their Virasoro descendants). The first few terms of this expansion are

$$\begin{aligned} g_+(\mathcal{T}^1,z_1)g_+(\mathcal{T}^2,z_2) = z_{12}^{\Delta^g(\mathcal{T}^3)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)} \Big\{ & \Big[1+\frac{1}{2}\sum_{r=1}^{\infty}\frac{z_{12}^r}{r!}\partial_2^r\Big] g_+(\mathcal{T}^3,z_2) \\ + \sum_{r=0}^{\infty}\frac{z_{12}^{r+1}}{(r+1)!}\partial_2^r \,\{2L_g^{ab} :g_+(\mathcal{T}^3,z_2)J_b(z_2):\} & \frac{1}{2}[\mathcal{T}_a^1-\mathcal{T}_a^2] + \ldots \Big\} \end{aligned} \tag{3.17}$$

and the omitted terms are of the form $:g_+J^p:$, $p\geq 2$ and derivatives thereof.

D. Braid relation. To discuss braiding, we will use the Euclidean continuation formula

$$(z-w) = (w-z)e^{i\pi \mathrm{sign}(\arg(z/w))} \tag{3.18}$$

where $|z| > |w|$ on the left and $|w| > |z|$ on the right. The phase in (3.18) is obtained by requiring agreement with the corresponding computations for vertex operators on the Minkowski world sheet (where z and w are on the unit circle).

The braid relation of two chiral vertex operators is then

$$g_+(\mathcal{T}^1, z_1)g_+(\mathcal{T}^2, z_2) = \mathcal{B}(\mathcal{T}^1\mathcal{T}^2)g_+(\mathcal{T}^2, z_2)g_+(\mathcal{T}^1, z_1) \tag{3.19a}$$

$$\mathcal{B}(\mathcal{T}^1\mathcal{T}^2) = e^{i\pi[\Delta^g(\mathcal{T}^3)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)]\mathrm{sign}(\arg(z_1/z_2))} \quad , \quad \mathcal{B}(\mathcal{T}^2\mathcal{T}^1) = \mathcal{B}^{-1}(\mathcal{T}^1\mathcal{T}^2) \tag{3.19b}$$

where $\mathcal{B}$ is the 1×1 braid matrix of the abelian theory.

E. Chiral correlators. The chiral correlators exhibit the Koba-Nielsen factor

$$A_g^+(\mathcal{T}, z) = \langle 0|G_+(\mathcal{T}^1)\cdots G_+(\mathcal{T}^n)|0\rangle \prod_{i<j}^n z_{ij}^{2L_g^{ab}\mathcal{T}_a^i\mathcal{T}_b^j} = \prod_{i<j} z_{ij}^{\Delta^g(\mathcal{T}^i+\mathcal{T}^j)-\Delta^g(\mathcal{T}^i)-\Delta^g(\mathcal{T}^j)}\delta(\sum_{i=1}^n \mathcal{T}^i) \tag{3.20}$$

where δ is Dirac delta function.

F. Antichiral sector. The antichiral vertex operators are obtained in the same way,

$$g_-(\mathcal{T}, \bar z) = \exp(-2iL_g^{ab}\mathcal{T}_a\bar Q_b^-(\bar z))G_-(\mathcal{T})\bar z^{-2L_g^{ab}\mathcal{T}_a\bar J_b(0)}\exp(-2iL_g^{ab}\mathcal{T}_a\bar Q_b^+(\bar z)) \tag{3.21a}$$

$$= \bar z^{-2\Delta^g(\mathcal{T})}\exp(-2iL_g^{ab}\mathcal{T}_a\bar Q_b^-(\bar z))\bar z^{-2L_g^{ab}\mathcal{T}_a\bar J_b(0)}\exp(-2iL_g^{ab}\mathcal{T}_a\bar Q_b^+(\bar z))G_-(\mathcal{T}) \tag{3.21b}$$

$$G_-(\mathcal{T}) = \mathrm{e}^{-i\bar q^a\mathcal{T}_a} \quad , \quad [\bar q^a, \bar J_b(m)] = i\delta_b^a\delta_{m,0} \tag{3.21c}$$

where (3.21b) is written with the antichiral zero mode $G_-(\mathcal{T})$ on the right. The intrinsic monodromy relations of the antichiral sector are

$$g_-(\mathcal{T}, \bar z e^{-2\pi i}) = e^{2\pi i[\Delta_g(\mathcal{T})+2L_g^{ab}\mathcal{T}_a\bar J_b(0)]}g_-(\mathcal{T}, \bar z) \tag{3.22a}$$

$$\begin{aligned}&\langle 0|g_-(\mathcal{T}^1, \bar z_1)\cdots g_-(\mathcal{T}^i, \bar z_i e^{-2\pi i})\cdots g_-(\mathcal{T}^n, \bar z_n)|0\rangle \\ &\quad = \mathrm{e}^{4\pi i L_g^{ab}\mathcal{T}_a^i\sum_{j\le i}^n\mathcal{T}_b^j}\langle 0|g_-(\mathcal{T}^1, \bar z_1)\cdots g_-(\mathcal{T}^i, \bar z_i)\cdots g_-(\mathcal{T}^n, \bar z_n)|0\rangle\end{aligned} \tag{3.22b}$$

and we note that the phases in (3.22b) are opposite to those of the chiral sector in (3.12). The OPE of two antichiral vertex operators and the antichiral correlators $A_g^-(\mathcal{T}, \bar z)$ may be obtained from (3.15) and (3.20) with $+ \to -$, $z \to \bar z$ and $\mathcal{T} \to -\mathcal{T}$.

G. Nonchiral results. Combining the chiral and antichiral vertex operators, we have the nonchiral vertex operators

$$g(\mathcal{T}, \bar z, z) = g_-(\mathcal{T}, \bar z)g_+(\mathcal{T}, z) \tag{3.23a}$$

$$\begin{aligned}&= \bar z^{-2\Delta^g(\mathcal{T})}\exp(-2iL_g^{ab}\mathcal{T}_a\bar Q_b^-(\bar z))\bar z^{-2L_g^{ab}\mathcal{T}_a\bar J_b(0)}\exp(-2iL_g^{ab}\mathcal{T}_a\bar Q_b^+(\bar z))G(\mathcal{T}) \\ &\quad \times \exp(2iL_g^{ab}\mathcal{T}_aQ_b^-(z))z^{2L_g^{ab}\mathcal{T}_aJ_b(0)}\exp(2iL_g^{ab}\mathcal{T}_aQ_b^+(z))\end{aligned} \tag{3.23b}$$

$$G(\mathcal{T}) \equiv G_-(\mathcal{T})G_+(\mathcal{T}) = \mathrm{e}^{i(q^a-\bar{q}^a)\mathcal{T}_a} \tag{3.23c}$$

where we have made the conventional assumption (which solves (2.20)) that q_a ($\bar{q}_a$) commutes with all the operators of the antichiral (chiral) sector. This assumption is examined in further detail for the nonabelian case in Section 7.

The nonchiral vertex operators (3.23) give the OPE's and nonchiral correlators

$$\begin{aligned}
&g(\mathcal{T}^1,\bar{z}_1,z_1)g(\mathcal{T}^2,\bar{z}_2,z_2)\\
&= |z_{12}|^{2[\Delta^g(\mathcal{T}^1+\mathcal{T}^2)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)]}\Big\{ g(\mathcal{T}^3,\bar{z}_2,z_2)\\
&+\sum_{r=0}^{\infty}\frac{z_{12}^{r+1}}{(r+1)!}2L_g^{ab} : g(\mathcal{T}^3,\bar{z}_2,z_2)\partial_2^r J_b(z_2) : \mathcal{T}_a^1\\
&-\sum_{r=0}^{\infty}\frac{\bar{z}_{12}^{r+1}}{(r+1)!}2L_g^{ab} : g(\mathcal{T}^3,\bar{z}_2,z_2)\bar{\partial}_2^r \bar{J}_b(\bar{z}_2) : \mathcal{T}_a^1\\
&+\sum_{r,s=0}^{\infty}\frac{z_{12}^{r+s+2}}{(r+s+2)!}4L_g^{ab}L_g^{cd} : g(\mathcal{T}^3,\bar{z}_2,z_2)\partial_2^r[\partial_2^s J_b(z_2)J_c(z_2)] : \mathcal{T}_d^1\mathcal{T}_a^1\\
&+\sum_{r,s=0}^{\infty}\frac{\bar{z}_{12}^{r+s+2}}{(r+s+2)!}4L_g^{ab}L_g^{cd} : g(\mathcal{T}^3,\bar{z}_2,z_2)\bar{\partial}_2^r[\bar{\partial}_2^s \bar{J}_b(\bar{z}_2)\bar{J}_c(\bar{z}_2)] : \mathcal{T}_a^1\mathcal{T}_d^1\\
&-\sum_{r,s=0}^{\infty}\frac{z_{12}^{r+1}}{(r+1)!}\frac{\bar{z}_{12}^{s+1}}{(s+1)!}4L_g^{ab}L_g^{cd} : g(\mathcal{T}^3,\bar{z}_2,z_2)\partial_2^r J_b(z_2)\bar{\partial}_2^s \bar{J}_c(\bar{z}_2) : \mathcal{T}_d^1\mathcal{T}_a^1\\
&\qquad + \text{higher affine secondaries}\Big\}
\end{aligned} \tag{3.24a}$$

$$A_g(\mathcal{T},\bar{z},z) = A_g^-(\mathcal{T},\bar{z})A_g^+(\mathcal{T},z) = \prod_{i<j}|z_{ij}|^{2[\Delta^g(\mathcal{T}^i+\mathcal{T}^j)-\Delta^g(\mathcal{T}^i)-\Delta^g(\mathcal{T}^j)]}\delta^2(\sum_{i=1}^{n}\mathcal{T}^i) \quad . \tag{3.24b}$$

Both of these results show trivial monodromy when any z is taken around another, and, moreover, the Virasoro-Shapiro factor in (3.24b) shows that the intrinsic monodromies (3.12) and (3.22b) have cancelled as they should.

4 Semiclassical Nonabelian Vertex Operators for the Affine-Sugawara Constructions

Chiral sector

The defining relations for the general chiral fields $g_+(\mathcal{T},z)$ are

$$g_+(\mathcal{T},z) = g_+(\mathcal{T},z_0) + \int_{z_0}^{z} \mathrm{d}z' \; 2L_g^{ab} : g_+(\mathcal{T},z')J_a(z') : \mathcal{T}_b \tag{4.1a}$$

$$[J_a(m), J_b(n)] = i f_{ab}{}^{c} J_c(m+n) + km\eta_{ab}\delta_{m+n,0} \tag{4.1b}$$

$$[J_a(m), g_+(\mathcal{T},z)] = g_+(\mathcal{T},z) z^m \mathcal{T}_a \quad , \quad [\mathcal{T}_a, \mathcal{T}_b] = i f_{ab}{}^{c}\mathcal{T}_c \tag{4.1c}$$

$$J_a(m \neq 0) = \mathcal{O}(k^{1/2}) \quad , \quad J_a(0) = \mathcal{O}(k^0) \tag{4.1d}$$

$$g_+(\mathcal{T},z) = \mathcal{O}(k^0) \quad , \quad \mathcal{T}_a = \mathcal{O}(k^0) \tag{4.1e}$$

$$L_g^{ab} = \mathcal{O}(k^{-1}) \quad , \quad \Delta_g(\mathcal{T}) = \mathcal{O}(k^{-1}) \tag{4.1f}$$

where z_0 is a regular reference point and L_g^{ab} is given in (2.6c). Using the results of the previous section as a guide, and paying close attention to the level-orders (4.1d-f), the iterative solution of (4.1a) can be rearranged into a z_0-independent semiclassical or high-level expansion of g_+.

We give here the solution of (4.1a) up to $\mathcal{O}(k^{-3/2})$:

$$\begin{aligned} g_+(\mathcal{T},z) &= G_+(\mathcal{T}) + 2iL_g^{ab}[Q_a^-(z)G_+(\mathcal{T}) + G_+(\mathcal{T})Q_a^+(z) - G_+(\mathcal{T})iJ_a(0)\ln z]\mathcal{T}_b \\ &+4L_g^{ab}L_g^{cd}\Bigg\{\sum_{m,n>0}[J_b(-m)\frac{J_c(-n)}{n}G_+(\mathcal{T})\frac{z^{m+n}}{m+n} + G_+(\mathcal{T})\frac{J_c(n)}{n}J_b(m)\frac{z^{-(m+n)}}{m+n}] \\ &- \sum_{\substack{m,n>0 \\ m\neq n}}[J_b(-m)G_+(\mathcal{T})\frac{J_c(n)}{n}\frac{z^{m-n}}{m-n} + \frac{J_c(-n)}{n}G_+(\mathcal{T})J_b(m)\frac{z^{-(m-n)}}{m-n}] \\ &+ \sum_{n>0}[\frac{J_c(-n)}{n}G_+(\mathcal{T})J_b(n)(\ln z - \frac{1}{2n}) - J_b(-n)G_+(\mathcal{T})\frac{J_c(n)}{n}(\ln z + \frac{1}{2n})]\Bigg\}\mathcal{T}_d\mathcal{T}_a \\ &+\mathcal{O}(k^{-3/2}) \quad . \end{aligned} \tag{4.2}$$

This is the explicit semiclassical form of the new nonabelian chiral vertex operators. Here, $G_+(\mathcal{T})$ is the constant chiral zero mode, which carries the index structure $G_+(\mathcal{T})_A{}^\alpha$ (in parallel with g_+), and which satisfies

$$G_+(\mathcal{T}) = \mathcal{O}(k^0) \tag{4.3a}$$

$$\partial G_+(\mathcal{T}) = 0 + \mathcal{O}(k^{-3/2}) \quad . \tag{4.3b}$$

Using the level-orders (4.1d-f) and (4.3a), one sees that the terms proportional to $Q_\pm$ in

(4.2) are $\mathcal{O}(k^{-1/2})$, while the rest of the explicit terms are $\mathcal{O}(k^{-1})$. Using (4.3b), it is straightforward to check by differentiation that the chiral vertex operator (4.2) satisfies

$$\partial g_+(\mathcal{T},z) = 2L_g^{ab} : g_+(\mathcal{T},z)J_a(z) : \mathcal{T}_b + \mathcal{O}(k^{-3/2}) \tag{4.4}$$

as it should.

The result (4.2) can be inverted to write the zero mode G_+ in terms of the primary field g_+,

$$\begin{aligned} G_+(\mathcal{T}) = g_+(\mathcal{T},z) - 2iL_g^{ab}[Q_a^-(z)g_+(\mathcal{T},z) + g_+(\mathcal{T},z)Q_a^+(z) - g_+(\mathcal{T},z)iJ_a(0)\ln z]\mathcal{T}_b \quad (4.5) \\ +4L_g^{ab}L_g^{cd}\Bigg\{\sum_{m,n>0}[\frac{J_b(-m)}{m}J_c(-n)g_+(\mathcal{T},z)\frac{z^{m+n}}{m+n} + g_+(\mathcal{T},z)J_c(n)\frac{J_b(m)}{m}\frac{z^{-(m+n)}}{m+n}] \\ + \sum_{\substack{m,n>0 \\ m\neq n}}[\frac{J_b(-m)}{m}g_+(\mathcal{T},z)J_c(n)\frac{z^{m-n}}{m-n} + J_c(-n)g_+(\mathcal{T},z)\frac{J_b(m)}{m}\frac{z^{-(m-n)}}{m-n}] \\ + \sum_{n>0}[J_b(-n)g_+(\mathcal{T},z)\frac{J_c(n)}{n}(\ln z - \frac{1}{2n}) - \frac{J_c(-n)}{n}g_+(\mathcal{T},z)J_b(n)(\ln z + \frac{1}{2n})]\Bigg\}\mathcal{T}_d\mathcal{T}_a \\ +\mathcal{O}(k^{-3/2}) \end{aligned}$$

and it is straightforward to check by differentiation with (4.4) that $G_+(\mathcal{T})$ in (4.5) satisfies (4.3b). Moreover, by setting $z = z_0$ in (4.2) and (4.5) one can see the intermediate relations between the zero mode $G_+(\mathcal{T})$ and the primary field $g_+(\mathcal{T},z_0)$ at the reference point z_0 (These relations are the nonabelian analogues of (3.4a) and (3.5)). The zero mode is of course independent of z_0

$$\partial_{z_0}G_+(\mathcal{T}) = 0 + \mathcal{O}(k^{-3/2}) \tag{4.6}$$

just as it is independent of z, because the differential equation (4.4) holds as well at the reference point.

Following the previous section, the next step is to use the inversion (4.5) and the algebra (4.1c) of the currents with the primary field g_+ to obtain the algebra of the currents with the zero mode G_+. After some algebra, the result is

$$[J_a(0), G_+(\mathcal{T})] = G_+(\mathcal{T})\mathcal{T}_a + \mathcal{O}(k^{-3/2}) \tag{4.7a}$$

$$[J_a(m\neq 0), G_+(\mathcal{T})] = \frac{i}{2km} : G_+(\mathcal{T})J_b(m) : f_a{}^{bc}\mathcal{T}_c + \mathcal{O}(k^{-1}) \tag{4.7b}$$

which reduces to the algebra (3.6) in the abelian case. It is likely that the relation (4.7a) is exact to all orders.

The algebra (4.7) shows that

$$[J_a(m\neq 0), G_+(\mathcal{T})] = \mathcal{O}(k^{-1/2}) \tag{4.8}$$

so, without loss of accuracy, we may move any factor G_+ in (4.2) thru any non-zero moded current. In particular, the chiral vertex operator may be written with the chiral zero mode on the left

$$g_+(\mathcal{T},z) = G_+(\mathcal{T})[1\!\!1 + iX^a(z)\mathcal{T}_a + N^{ab}(z)\mathcal{T}_b\mathcal{T}_a] + \mathcal{O}(k^{-3/2}) \tag{4.9}$$

where we have defined the quantities

$$X^a(z) \equiv 2L_g^{ab}[Q_b^-(z) + Q_b^+(z) - iJ_b(0)\ln z] \tag{4.10a}$$

$$\begin{aligned} N^{ab}(z) \equiv 4L_g^{ac}L_g^{db}\Bigg\{ &\sum_{m,n>0}[J_c(-m)\frac{J_d(-n)}{n}\frac{z^{m+n}}{m+n} + \frac{J_d(n)}{n}J_c(m)\frac{z^{-(m+n)}}{m+n}] \\ &- \sum_{\substack{m,n>0 \\ m\neq n}}[J_c(-m)\frac{J_d(n)}{n}\frac{z^{m-n}}{m-n} + \frac{J_d(-n)}{n}J_c(m)\frac{z^{-(m-n)}}{m-n}] \\ &+ \sum_{n>0}[\frac{J_d(-n)}{n}J_c(n)(\ln z - \frac{1}{2n}) - J_c(-n)\frac{J_d(n)}{n}(\ln z + \frac{1}{2n})]\Bigg\} \quad . \end{aligned} \tag{4.10b}$$

It will also be convenient to have the inversion of (4.9)

$$G_+(\mathcal{T}) = g_+(\mathcal{T},z)[1\!\!1 - iX^a(z)\mathcal{T}_a - (N^{ab}(z) + X^b(z)X^a(z))\mathcal{T}_b\mathcal{T}_a] + \mathcal{O}(k^{-3/2}) \tag{4.11}$$

which agrees with eq.(4.5) thru the indicated order.

Restoring the Lie algebra and quantum group indices α and A,

$$g_+(\mathcal{T},z)_A{}^\alpha = G_+(\mathcal{T})_A{}^\beta[1\!\!1 + iX^a(z)\mathcal{T}_a + N^{ab}(z)\mathcal{T}_b\mathcal{T}_a]_\beta{}^\alpha + \mathcal{O}(k^{-3/2}) \tag{4.12a}$$

$$\alpha, A = 1 \ldots \dim \mathcal{T} \tag{4.12b}$$

we see that, thru this order of the semiclassical expansion, the quantum group acts only on the chiral zero mode G_+.

We also remark that the algebra (4.7) is consistent with unitarity[a] of the chiral zero mode

$$G_+^\dagger(\mathcal{T})G_+(\mathcal{T}) = G_+(\mathcal{T})G_+^\dagger(\mathcal{T}) = 1\!\!1 + \mathcal{O}(k^{-1}) \tag{4.13}$$

which implies that the extreme semiclassical chiral vertex operator

$$g_+(\mathcal{T},z) = G_+(\mathcal{T})\exp[2iL_g^{ab}(Q_a^-(z) + Q_a^+(z))\mathcal{T}_b] + \mathcal{O}(k^{-1}) \tag{4.14}$$

is also unitary

$$g_+^\dagger(\mathcal{T},z)g_+(\mathcal{T},z) = g_+(\mathcal{T},z)g_+^\dagger(\mathcal{T},z) = 1\!\!1 + \mathcal{O}(k^{-1}) \tag{4.15}$$

[a]Unitarity is easiest to check on the unit circle (where $z^* = z^{-1}$) using a Cartesian frame with $\eta_{ab} = \delta_{ab}$, $J_a^\dagger(m) = J_a(-m)$ and $\mathcal{T}_a$ Hermitean.

thru the indicated order. It is known from the abelian case that vertex operators cannot be unitary operators beyond this order, due to normal ordering, which enters in g_+ (and G_+) at order k^{-1}.

Antichiral sector

Following similar steps, we have solved the antichiral ODE

$$\bar{\partial} g_-(\mathcal{T},\bar{z}) = -2L_g^{ab}\mathcal{T}_b : g_-(\mathcal{T},\bar{z})\bar{J}_a(\bar{z}) : +\mathcal{O}(k^{-3/2}) \tag{4.16}$$

for the antichiral vertex operator $g_-(\mathcal{T},\bar{z})$ thru the same order. The main results are as follows:

1. Antichiral vertex operator and zero mode. The antichiral vertex operator is

$$\begin{aligned} g_-(\mathcal{T},\bar{z}) = {} & G_-(\mathcal{T}) - 2iL_g^{ab}\mathcal{T}_b[\bar{Q}_a^-(\bar{z})G_-(\mathcal{T}) + G_-(\mathcal{T})\bar{Q}_a^+(\bar{z}) - G_-(\mathcal{T})i\bar{J}_a(0)\ln\bar{z}] \\ & +4L_g^{ab}L_g^{cd}\mathcal{T}_a\mathcal{T}_d\Bigg\{\sum_{m,n>0}[\bar{J}_b(-m)\frac{\bar{J}_c(-n)}{n}G_-(\mathcal{T})\frac{\bar{z}^{m+n}}{m+n} + G_-(\mathcal{T})\frac{\bar{J}_c(n)}{n}\bar{J}_b(m)\frac{\bar{z}^{-(m+n)}}{m+n}] \\ & - \sum_{\substack{m,n>0\\ m\neq n}}[\bar{J}_b(-m)G_-(\mathcal{T})\frac{\bar{J}_c(n)}{n}\frac{\bar{z}^{m-n}}{m-n} + \frac{\bar{J}_c(-n)}{n}G_-(\mathcal{T})\bar{J}_b(m)\frac{\bar{z}^{-(m-n)}}{m-n}] \\ & + \sum_{n>0}[\frac{\bar{J}_c(-n)}{n}G_-(\mathcal{T})\bar{J}_b(n)(\ln\bar{z} - \frac{1}{2n}) - \bar{J}_b(-n)G_-(\mathcal{T})\frac{\bar{J}_c(n)}{n}(\ln\bar{z} + \frac{1}{2n})]\Bigg\} \\ & +\mathcal{O}(k^{-3/2}) \end{aligned} \tag{4.17}$$

where G_-, with index structure $G_-(\mathcal{T})_\alpha{}^A$, is the antichiral zero mode:

$$G_-(\mathcal{T}) = \mathcal{O}(k^0) \tag{4.18a}$$

$$\bar{\partial} G_-(\mathcal{T}) = 0 + \mathcal{O}(k^{-3/2}) \quad . \tag{4.18b}$$

Inversion of (4.17) gives the antichiral zero mode G_- in terms of the antichiral primary field g_-

$$\begin{aligned} G_-(\mathcal{T}) = {} & g_-(\mathcal{T},\bar{z}) + 2iL_g^{ab}\mathcal{T}_b[\bar{Q}_a^-(\bar{z})g_-(\mathcal{T},\bar{z}) + g_-(\mathcal{T},\bar{z})\bar{Q}_a^+(\bar{z}) - g_-(\mathcal{T},\bar{z})i\bar{J}_a(0)\ln\bar{z}] \\ & +4L_g^{ab}L_g^{cd}\mathcal{T}_a\mathcal{T}_d\Bigg\{\sum_{m,n>0}[\frac{\bar{J}_b(-m)}{m}\bar{J}_c(-n)g_-(\mathcal{T},\bar{z})\frac{\bar{z}^{m+n}}{m+n} + g_-(\mathcal{T},\bar{z})\bar{J}_c(n)\frac{\bar{J}_b(m)}{m}\frac{\bar{z}^{-(m+n)}}{m+n}] \\ & + \sum_{\substack{m,n>0\\ m\neq n}}[\frac{\bar{J}_b(-m)}{m}g_-(\mathcal{T},\bar{z})\bar{J}_c(n)\frac{\bar{z}^{m-n}}{m-n} + \bar{J}_c(-n)g_-(\mathcal{T},\bar{z})\frac{\bar{J}_b(m)}{m}\frac{\bar{z}^{-(m-n)}}{m-n}] \\ & + \sum_{n>0}[\bar{J}_b(-n)g_-(\mathcal{T},\bar{z})\frac{\bar{J}_c(n)}{n}(\ln\bar{z} - \frac{1}{2n}) - \frac{\bar{J}_c(-n)}{n}g_-(\mathcal{T},\bar{z})\bar{J}_b(n)(\ln\bar{z} + \frac{1}{2n})]\Bigg\} \\ & +\mathcal{O}(k^{-3/2}) \end{aligned} \tag{4.19}$$

and it is not difficult to check that the results (4.17) and (4.19) satisfy the differential equations (4.16) and (4.18b).

2. Algebra of the zero modes. Using (4.19) and the algebra (2.19b) of the antichiral currents with the primary field g_-, we obtain the algebra of the antichiral currents with the antichiral zero mode G_-,

$$[\bar{J}_a(0), G_-(\mathcal{T})] = -\mathcal{T}_a G_-(\mathcal{T}) + \mathcal{O}(k^{-3/2}) \tag{4.20a}$$

$$[\bar{J}_a(m \neq 0), G_-(\mathcal{T})] = -\frac{i}{2km} f_a{}^{bc} \mathcal{T}_c : G_-(\mathcal{T}) \bar{J}_b(m) : + \mathcal{O}(k^{-1}) \tag{4.20b}$$

so that $[\bar{J}_a(m \neq 0), G_-(\mathcal{T})] = \mathcal{O}(k^{-1/2})$ as in the chiral sector.

3. Rearrangements. Using the algebra (4.20), a number of alternate forms may be obtained for the antichiral vertex operator,

$$g_-(\mathcal{T}, \bar{z}) = G_-(\mathcal{T}) - i\mathcal{T}_a G_-(\mathcal{T}) \bar{X}^a(\bar{z}) + \mathcal{T}_a \mathcal{T}_b G_-(\mathcal{T}) \bar{N}^{ab}(\bar{z}) + \mathcal{O}(k^{-3/2}) \tag{4.21a}$$

$$= [\mathbb{1} - i\mathcal{T}_a \bar{X}^a(\bar{z}) - 2\Delta^g(\mathcal{T}) \ln \bar{z} + \mathcal{T}_a \mathcal{T}_b \bar{N}^{ab}(\bar{z})] G_-(\mathcal{T}) + \mathcal{O}(k^{-3/2}) \tag{4.21b}$$

$$= \bar{z}^{-2\Delta^g(\mathcal{T})} [\mathbb{1} - i\mathcal{T}_a \bar{X}^a(\bar{z}) + \mathcal{T}_a \mathcal{T}_b \bar{N}^{ab}(\bar{z})] G_-(\mathcal{T}) + \mathcal{O}(k^{-3/2}) \tag{4.21c}$$

$$\bar{X}^a(\bar{z}) \equiv X^a(z)|_{z \to \bar{z}, J \to \bar{J}} \quad , \quad \bar{N}^{ab}(\bar{z}) \equiv N^{ab}(z)|_{z \to \bar{z}, J \to \bar{J}} \quad . \tag{4.21d}$$

where X and N are defined in (4.10a,b). It will also be useful to have the inverse of (4.21c),

$$G_-(\mathcal{T}) = \bar{z}^{2\Delta^g(\mathcal{T})} [\mathbb{1} + i\mathcal{T}_a \bar{X}^a(\bar{z}) - \mathcal{T}_a \mathcal{T}_b (\bar{N}^{ab}(\bar{z}) + \bar{X}^a(\bar{z}) \bar{X}^b(\bar{z}))] g_-(\mathcal{T}, \bar{z}) + \mathcal{O}(k^{-3/2}) \tag{4.22}$$

which agrees with (4.19) thru the indicated order.

In (4.21b,c) we have chosen to write the antichiral zero mode $G_-(\mathcal{T})$ on the right. The index structure of G_- is $(G_-)_\alpha{}^A$, so these forms show that the quantum group acts only on G_- thru this order of the semiclassical expansion.

4. Semiclassical unitarity. As in the chiral sector, the algebra (4.20) is consistent with semiclassical unitarity of the antichiral zero mode

$$G_-^\dagger(\mathcal{T}) G_-(\mathcal{T}) = G_-(\mathcal{T}) G_-^\dagger(\mathcal{T}) = \mathbb{1} + \mathcal{O}(k^{-1}) \tag{4.23}$$

and this implies that the extreme semiclassical antichiral vertex operator

$$g_-(\mathcal{T}, \bar{z}) = \exp[-2iL_g^{ab}(\bar{Q}_a^-(\bar{z}) + \bar{Q}_a^+(\bar{z}))\mathcal{T}_b] G_-(\mathcal{T}) + \mathcal{O}(k^{-1}) \tag{4.24}$$

is also unitary

$$g_-^\dagger(\mathcal{T}, \bar{z}) g_-(\mathcal{T}, \bar{z}) = g_-(\mathcal{T}, \bar{z}) g_-^\dagger(\mathcal{T}, \bar{z}) = \mathbb{1} + \mathcal{O}(k^{-1}) \tag{4.25}$$

thru the indicated order.

Some simple chiral and antichiral applications

We conclude this section with some simple applications of these results.

A. Affine primary fields. By construction, the chiral and antichiral vertex operators (4.9) and (4.21c) are ($\dim \mathcal{T}$ quantum group "copies" of) affine-primary fields under their respective affine algebras,

$$\left.\begin{array}{r} [J_a(m), g_+(\mathcal{T},z)_A{}^\alpha] = g_+(\mathcal{T},z)_A{}^\beta z^m (\mathcal{T}_a)_\beta{}^\alpha \\ [\bar{J}_a(m), g_-(\mathcal{T},\bar{z})_\alpha{}^A] = -\bar{z}^m (\mathcal{T}_a)_\alpha{}^\beta g_-(\mathcal{T},\bar{z})_\beta{}^A \end{array}\right\} + \left\{\begin{array}{ll} \mathcal{O}(k^{-3/2}) & \text{when } m=0 \\ \mathcal{O}(k^{-1}) & \text{when } m\neq 0 \end{array}\right. . \tag{4.26}$$

These relations can also be checked directly using the current algebra and the algebra (4.7), (4.20) of the currents with the zero modes $G_\pm$. On the affine vacuum, the vertex operators create (copies of) affine-primary states

$$g_+(\mathcal{T},0)_A{}^\alpha |0\rangle = G_+(\mathcal{T})_A{}^\alpha |0\rangle + \mathcal{O}(k^{-3/2}) \tag{4.27a}$$

$$g_-(\mathcal{T},0)_\alpha{}^A |0\rangle = G_-(\mathcal{T})_\alpha{}^A |0\rangle + \mathcal{O}(k^{-3/2}) \tag{4.27b}$$

which, as in the abelian case, are proportional to the chiral and antichiral zero modes.

B. Affine-Sugawara primary fields. We have also checked explicitly that

$$T_g(z) g_+(\mathcal{T},w) = \left[\frac{\Delta_g(\mathcal{T})}{(z-w)^2} + \frac{1}{z-w}\partial_w\right] g_+(\mathcal{T},w) + \mathcal{O}(k^{-3/2}) \tag{4.28a}$$

$$\bar{T}_g(\bar{z}) g_-(\mathcal{T},\bar{w}) = \left[\frac{\Delta_g(\mathcal{T})}{(\bar{z}-\bar{w})^2} + \frac{1}{\bar{z}-\bar{w}}\partial_w\right] g_-(\mathcal{T},\bar{w}) + \mathcal{O}(k^{-3/2}) \tag{4.28b}$$

so that, as they should be, the chiral and antichiral vertex operators are Virasoro primary fields under the affine-Sugawara constructions T_g and $\bar{T}_g$ respectively.

C. Intrinsic monodromies. When z is taken around a closed loop, one finds the intrinsic monodromy relations,

$$\begin{aligned} g_+(\mathcal{T}, ze^{2\pi i}) = g_+(\mathcal{T},z) \exp\left(4\pi i [L_g^{ab}\mathcal{T}_a J_b(0) + 2i L_g^{ac} L_g^{db} f_{ab}{}^e \mathcal{T}_e \sum_{n>0} \frac{J_c(-n)J_d(n)}{n}]\right) \\ + \mathcal{O}(k^{-3/2}) \end{aligned} \tag{4.29a}$$

$$\begin{aligned} g_-(\mathcal{T}, \bar{z}e^{-2\pi i}) = \exp\left(4\pi i [\Delta^g(\mathcal{T}) + L_g^{ab}\mathcal{T}_a \bar{J}_b(0) + 2i L_g^{ac} L_g^{db} f_{ab}{}^e \mathcal{T}_e \sum_{n>0} \frac{\bar{J}_c(-n)\bar{J}_d(n)}{n}]\right) \\ \times g_-(\mathcal{T},\bar{z}) + \mathcal{O}(k^{-3/2}) \quad . \end{aligned} \tag{4.29b}$$

Using (2.2), (4.7), (4.20) and the $(g\times g)$-global Ward identities, the operator relations (4.29) imply the much simpler monodromies for the correlators

$$\begin{aligned} &\langle 0| g_+(\mathcal{T}^1, z_1) \cdots g_+(\mathcal{T}^i, z_i e^{2\pi i}) \cdots g_+(\mathcal{T}^n, z_n)|0\rangle \\ &= \langle 0| g_+(\mathcal{T}^1, z_1) \cdots g_+(\mathcal{T}^i, z_i) \cdots g_+(\mathcal{T}^n, z_n)|0\rangle e^{-4\pi i L_g^{ab}\mathcal{T}_a^i \sum_{j\leq i}^n \mathcal{T}_b^j} + \mathcal{O}(k^{-3/2}) \end{aligned} \tag{4.30a}$$

$$
\begin{aligned}
&\langle 0|g_-(\mathcal{T}^1,\bar z_1)\cdots g_-(\mathcal{T}^i,\bar z_i e^{-2\pi i})\cdots g_-(\mathcal{T}^n,\bar z_n)|0\rangle \\
&\quad = e^{4\pi i L_g^{ab}\mathcal{T}_a^i \sum_{j\le i}^n \mathcal{T}_b^j}\langle 0|g_-(\mathcal{T}^1,\bar z_1)\cdots g_-(\mathcal{T}^i,\bar z_i)\cdots g_-(\mathcal{T}^n,\bar z_n)|0\rangle + \mathcal{O}(k^{-3/2})
\end{aligned} \tag{4.30b}
$$

because the normal-ordered terms in (4.29) do not contribute to the correlators at the indicated order.

5 Operator Products and Expansions

In this section, we combine our results above with those of Ref.[25] to obtain the chiral-chiral and antichiral-antichiral semiclassical operator products and OPE's of the semiclassical vertex operators (4.9) and (4.21c). The corresponding chiral-antichiral products and expansions are discussed in Section 7.

Chiral sector

The product of two chiral vertex operators (4.9) can be written as

$$
\begin{aligned}
g_+(\mathcal{T}^1,z_1)g_+(\mathcal{T}^2,z_2) &= :g_+(\mathcal{T}^1,z_1)g_+(\mathcal{T}^2,z_2):(1\!\!1 + 2L_g^{ab}\mathcal{T}_a^1\mathcal{T}_b^2\ln z_{12}) + \mathcal{O}(k^{-3/2}) \\
&= :g_+(\mathcal{T}^1,z_1)g_+(\mathcal{T}^2,z_2): z_{12}^{2L_g^{ab}\mathcal{T}_a^1\mathcal{T}_b^2} + \mathcal{O}(k^{-3/2})
\end{aligned} \tag{5.1}
$$

where the normal-ordered product of two vertex operators is defined in parallel to (3.13b),

$$
\begin{aligned}
:g_+(\mathcal{T}^1,z_1)g_+(\mathcal{T}^2,z_2): \equiv\ & G_+(\mathcal{T}^1)G_+(\mathcal{T}^2)[1\!\!1 + iX^a(z_1)\mathcal{T}_a^1 + iX^a(z_2)\mathcal{T}_a^2 \\
&+ N^{ab}(z_1)\mathcal{T}_b^1\mathcal{T}_a^1 + N^{ab}(z_2)\mathcal{T}_b^2\mathcal{T}_a^2 + N_2^{ab}(z_1,z_2)\mathcal{T}_a^1\mathcal{T}_b^2] + \mathcal{O}(k^{-3/2})
\end{aligned} \tag{5.2a}
$$

$$
\begin{aligned}
N_2^{ab}(z_1,z_2) \equiv&\ -:X^a(z_1)X^b(z_2): \\
=&-4L_g^{ac}L_g^{db}[Q_c^-(z_1)(Q_d^-(z_2)+Q_d^+(z_2)) + (Q_d^-(z_2)+Q_d^+(z_2))Q_c^+(z_1)] + \mathcal{O}(k^{-3/2})
\end{aligned} \tag{5.2b}
$$

with zero modes on the left.

For high-level closure of the affine-primary fields, the product of two zero modes must close into zero modes,

$$
G_+(\mathcal{T}^1)_{A_1}{}^{\alpha_1}G_+(\mathcal{T}^2)_{A_2}{}^{\alpha_2} = \sum_{\mathcal{T}_k,A_k,\alpha_k} F_{A_1A_2\alpha_k}^{\alpha_1\alpha_2A_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k)G_+(\mathcal{T}^k)_{A_k}{}^{\alpha_k} + \mathcal{O}(k^{-3/2}) \tag{5.3}
$$

which is the non-abelian analogue of (3.14). Following Ref.[25], we will assume that the fusion coefficient F in (5.3) has the factorized form

$$
F_{A_1A_2\alpha_k}^{\alpha_1\alpha_2A_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) = Q_{A_1A_2}{}^{A_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k)C^{\alpha_1\alpha_2}{}_{\alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) + \mathcal{O}(k^{-3/2}) \tag{5.4}
$$

Here C is the usual Clebsch-Gordon coefficient for $\mathcal{T}^1\otimes\mathcal{T}^2$ into $\mathcal{T}^k$, which satisfies the g-global Ward identity

$$
C^{\beta_1\beta_2}{}_{\alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k)[(\mathcal{T}_a^1)_{\beta_1}{}^{\alpha_1}\delta_{\beta_2}{}^{\alpha_2} + \delta_{\beta_1}{}^{\alpha_1}(\mathcal{T}_a^2)_{\beta_2}{}^{\alpha_2}] = (\mathcal{T}_a^k)_{\alpha_k}{}^{\beta_k}C^{\alpha_1\alpha_2}{}_{\beta_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) \tag{5.5}
$$

while Q is the corresponding (level-dependent) quantum Clebsch-Gordan coefficient.

Using (5.3) and (5.5), we may expand the operator product (5.2a) for z_1 near z_2. After some algebra, one finds the OPE of two chiral vertex operators

$$\begin{aligned}
& g_+(\mathcal{T}^1, z_1)_{A_1}{}^{\alpha_1} g_+(\mathcal{T}^2, z_2)_{A_2}{}^{\alpha_2} \\
&\quad = \sum_{\substack{\mathcal{T}^k, A_k, \alpha_k \\ \beta_1}} z_{12}^{\Delta^g(\mathcal{T}^k)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)} F^{\beta_1\alpha_2 A_k}_{A_1 A_2 \alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) \Big\{ g_+(\mathcal{T}^k, z_2)_{A_k}{}^{\alpha_k} \delta^{\alpha_1}_{\beta_1} \\
&\quad + \sum_{r=0}^{\infty} \frac{z_{12}^{r+1}}{(r+1)!} 2L_g^{ab} : g_+(\mathcal{T}^k, z_2)_{A_k}{}^{\alpha_k} \partial_2^r J_b(z_2) : (\mathcal{T}_a^1)_{\beta_1}{}^{\alpha_1} \\
&\quad + \sum_{r,s=0}^{\infty} \frac{z_{12}^{r+s+2}}{(r+s+2)!}\, 4L_g^{ab} L_g^{cd} : g_+(\mathcal{T}^k, z_2)_{A_k}{}^{\alpha_k} \partial_2^r [\partial_2^s J_b(z_2) J_c(z_2)] : (\mathcal{T}_d^1 \mathcal{T}_a^1)_{\beta_1}{}^{\alpha_1} \Big\} \\
&\quad + \mathcal{O}(k^{-3/2})
\end{aligned} \tag{5.6}$$

where the normal-ordered product $: g_+ J :$ is given in (2.10a), and

$$\begin{aligned}
: g_+(\mathcal{T}, z) J_a(z) J_b(z) : \equiv\ & J_a^-(z)[J_b^-(z) g_+(\mathcal{T}, z) + g_+(\mathcal{T}, z) J_b^+(z)] \\
& + [J_b^-(z) g_+(\mathcal{T}, z) + g_+(\mathcal{T}, z) J_b^+(z)] J_a^+(z) \quad .
\end{aligned} \tag{5.7}$$

The definition (5.7) can be replaced with (3.16) to the order we are working, since the extra current zero-mode contributions in (3.16) would contribute to (5.6) at $\mathcal{O}(k^{-3/2})$.

The right side of the OPE (5.6) shows the affine-primary fields g_+ and an infinite number of affine-secondary fields of the form $: g_+ J :$, $: g_+ J J :$ and derivatives thereof. One also notes that the chiral OPE has the schematic form [42]

$$\begin{aligned}
\text{affine-primary} \cdot \text{affine-primary} =\ & \mathcal{O}(k^0) \cdot \text{affine primaries} \\
& + \mathcal{O}(k^{-1}) \cdot \text{affine secondaries}
\end{aligned} \tag{5.8}$$

so that the affine-primary fields close into themselves in the extreme classical limit.

Using also the high-level ODE (4.4), one may rearrange (5.6) in the form analogous to eq.(3.17)

$$\begin{aligned}
& g_+(\mathcal{T}^1, z_1)_{A_1}{}^{\alpha_1} g_+(\mathcal{T}^2, z_2)_{A_2}{}^{\alpha_2} \\
& = \sum_{\substack{\mathcal{T}^k, A_k, \alpha_k \\ \beta_1, \beta_2}} z_{12}^{\Delta^g(\mathcal{T}^k)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)} F^{\beta_1\beta_2 A_k}_{A_1 A_2 \alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) \Big\{ \Big[1 + \frac{1}{2} \sum_{r=1}^{\infty} \frac{z_{12}^r}{r!} \partial_2^r \Big] g_+(\mathcal{T}^k, z_2)_{A_k}{}^{\alpha_k} \delta^{\alpha_1}_{\beta_1} \delta^{\alpha_2}_{\beta_2} \\
& \quad + \sum_{r=0}^{\infty} \frac{z_{12}^{r+1}}{(r+1)!} \partial_2^r \{ 2L_g^{ab} : g_+(\mathcal{T}^k, z_2)_{A_k}{}^{\alpha_k} J_b(z_2) : \} \frac{1}{2} [(\mathcal{T}_a^1)_{\beta_1}{}^{\alpha_1} \delta^{\alpha_2}_{\beta_2} - (\mathcal{T}_a^2)_{\beta_2}{}^{\alpha_2} \delta^{\alpha_1}_{\beta_1}] \Big\} \\
& \qquad + \mathcal{O}(k^{-1})
\end{aligned} \tag{5.9}$$

which groups the Virasoro primary fields with their Virasoro descendants. For simplicity, we have omitted the $\mathcal{O}(k^{-1})$ terms which are proportional to $: g_+ JJ :$ and derivatives thereof.

Antichiral sector

Following the same development for the product of two antichiral vertex operators, we find:

$$g_-(\mathcal{T}^1,\bar{z}_1)g_-(\mathcal{T}^2,\bar{z}_2) = \bar{z}_{12}^{2L_g^{ab}\mathcal{T}_a^1\mathcal{T}_b^2} : g_-(\mathcal{T}^1,\bar{z}_1)g_-(\mathcal{T}^2,\bar{z}_2) : +\mathcal{O}(k^{-3/2}) \tag{5.10a}$$

$$\begin{aligned} : g_-(\mathcal{T}^1,\bar{z}_1)g_-(\mathcal{T}^2,\bar{z}_2) : &\equiv G_-(\mathcal{T}^1)G_-(\mathcal{T}^2) \\ &-i\mathcal{T}_a^1 G_-(\mathcal{T}^1)G_-(\mathcal{T}^2)\bar{X}^a(\bar{z}_1) - i\mathcal{T}_a^2 G_-(\mathcal{T}^1)G_-(\mathcal{T}^2)\bar{X}^a(\bar{z}_2) \\ &+\mathcal{T}_a^1\mathcal{T}_b^1 G_-(\mathcal{T}^1)G_-(\mathcal{T}^2)\bar{N}^{ab}(\bar{z}_1) + \mathcal{T}_a^2\mathcal{T}_b^2 G_-(\mathcal{T}^1)G_-(\mathcal{T}^2)\bar{N}^{ab}(\bar{z}_2) \\ &+ \mathcal{T}_a^1\mathcal{T}_b^2 G_-(\mathcal{T}^1)G_-(\mathcal{T}^2)\bar{N}_2^{ab}(\bar{z}_1,\bar{z}_2) + \mathcal{O}(k^{-3/2}) \end{aligned} \tag{5.10b}$$

$$\begin{aligned} &= \bar{z}_1^{-2\Delta^g(\mathcal{T}^1)}\bar{z}_2^{-2\Delta^g(\mathcal{T}^2)}(\bar{z}_1\bar{z}_2)^{-2L_g^{ab}\mathcal{T}_a^1\mathcal{T}_b^2} \\ &\quad\times[\mathbb{1} - i\mathcal{T}_a^1\bar{X}^a(\bar{z}_1) - i\mathcal{T}_a^2\bar{X}^a(\bar{z}_2) \\ &\quad+ \mathcal{T}_a^1\mathcal{T}_b^1\bar{N}^{ab}(\bar{z}_1) + \mathcal{T}_a^2\mathcal{T}_b^2\bar{N}^{ab}(\bar{z}_2) \\ &\quad+ \mathcal{T}_a^1\mathcal{T}_b^2\bar{N}_2^{ab}(\bar{z}_1,\bar{z}_2)]G_-(\mathcal{T}^1)G_-(\mathcal{T}^2) + \mathcal{O}(k^{-3/2}) \end{aligned} \tag{5.10c}$$

$$\bar{N}_2^{ab}(\bar{z}_1,\bar{z}_2) \equiv N_2^{ab}(z_1,z_2)|_{z_1\to\bar{z}_1,z_2\to\bar{z}_2,J\to\bar{J}} \tag{5.10d}$$

$$G_-(\mathcal{T}^1)_{\alpha_1}{}^{A_1}G_-(\mathcal{T}^2)_{\alpha_2}{}^{A_2} = \sum_{\mathcal{T}^k,A_k,\alpha_k} \bar{F}_{\alpha_1\alpha_2 A_k}^{A_1A_2\alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k)G_-(\mathcal{T}^k)_{\alpha_k}{}^{A_k} + \mathcal{O}(k^{-3/2}) \tag{5.10e}$$

$$\bar{F}_{\alpha_1\alpha_2 A_k}^{A_1A_2\alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) = \bar{Q}^{A_1A_2}{}_{A_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k)\bar{C}_{\alpha_1\alpha_2}{}^{\alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) + \mathcal{O}(k^{-3/2}) \tag{5.10f}$$

$$[(\mathcal{T}_a^1)_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2} + \delta_{\alpha_1}{}^{\beta_1}(\mathcal{T}_a^2)_{\alpha_2}{}^{\beta_2}]\bar{C}_{\beta_1\beta_2}{}^{\alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) = \bar{C}_{\alpha_1\alpha_2}{}^{\beta_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k)(\mathcal{T}_a^k)_{\beta_k}{}^{\alpha_k} \tag{5.10g}$$

$$C^{\alpha\bar{\beta}}{}_{\cdot}(\mathcal{T}\bar{\mathcal{T}}\mathcal{T}_{(1)}) = \frac{1}{\sqrt{\dim\mathcal{T}}}\eta^{\alpha\bar{\beta}}(\mathcal{T}) \tag{5.10h}$$

$$Q_{A\bar{B}}{}^{\cdot}(\mathcal{T}\bar{\mathcal{T}}\mathcal{T}_{(1)}) = \frac{1}{\sqrt{\mathrm{Tr}\,\Lambda(\mathcal{T})}}\Lambda_{A\bar{B}}(\mathcal{T}) \tag{5.10i}$$

$$\bar{C}_{\alpha_1\alpha_2}{}^{\alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) = \eta_{\alpha_1\beta_1}(\mathcal{T}^1)\eta_{\alpha_2\beta_2}(\mathcal{T}^2)C^{\beta_1\beta_2}{}_{\beta_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k)^*\eta^{\beta_k\alpha_k}(\mathcal{T}^k) \tag{5.10j}$$

$$\bar{Q}^{A_1A_2}{}_{A_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) = \Lambda^{A_1B_1}(\mathcal{T}^1)\Lambda^{A_2B_2}(\mathcal{T}^2)Q_{B_1B_2}{}^{B_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k)^*\Lambda_{B_kA_k}(\mathcal{T}^k) \tag{5.10k}$$

$$\sum_{A_1A_2}\bar{Q}^{A_1A_2}{}_{A_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k)Q_{A_1A_2}{}^{A_l}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^l) = \delta_{\mathcal{T}^k,\mathcal{T}^l}\delta_{A_k}^{A_l} + \mathcal{O}(k^{-3/2}) \quad . \tag{5.10l}$$

Here $\mathcal{T}_{(1)}$ is the trivial representation, $\eta_{\alpha\beta}(\mathcal{T})$ is the carrier space metric of irrep $\mathcal{T}$ and $\Lambda_{AB}(\mathcal{T})$ is the corresponding invariant form on the quantum group. $\bar{C}$ and $\bar{Q}$ are the duals of the classical and quantum Clebsch-Gordan coefficients C and Q.

After some algebra, we then obtain the OPE of two antichiral vertex operators,

$$
\begin{aligned}
& g_-(\mathcal{T}^1,\bar z_1)_{\alpha_1}{}^{A_1} g_-(\mathcal{T}^2,\bar z_2)_{\alpha_2}{}^{A_2} \\
&\quad = \sum_{\substack{\mathcal{T}^k,A_k,\alpha_k\\ \beta_1}} \bar z_{12}^{\Delta^g(\mathcal{T}^k)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)} \Big\{ g_-(\mathcal{T}^k,\bar z_2)_{\alpha_k}{}^{A_k}\delta_{\alpha_1}^{\beta_1} \\
&\quad - \sum_{r=0}^{\infty} \frac{\bar z_{12}^{r+1}}{(r+1)!} 2L_g^{ab} : g_-(\mathcal{T}^k,\bar z_2)_{\alpha_k}{}^{A_k} \bar\partial_2^r \bar J_b(\bar z_2) : (\mathcal{T}_a^1)_{\alpha_1}{}^{\beta_1} \\
&\quad + \sum_{r,s=0}^{\infty} \frac{\bar z_{12}^{r+s+2}}{(r+s+2)!} 4L_g^{ab}L_g^{cd} : g_-(\mathcal{T}^k,\bar z_2)_{\alpha_k}{}^{A_k} \bar\partial_2^r[\bar\partial_2^s \bar J_b(z_2)\bar J_c(z_2)] : (\mathcal{T}_a^1\mathcal{T}_d^1)_{\alpha_1}{}^{\beta_1} \Big\} \\
&\qquad\qquad \times \bar F_{\beta_1\alpha_2 A_k}^{A_1A_2\alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) + \mathcal{O}(k^{-3/2})
\end{aligned}
\tag{5.11}
$$

and the antichiral analogue of (5.9)

$$
\begin{aligned}
& g_-(\mathcal{T}^1,\bar z_1)_{\alpha_1}{}^{A_1} g_-(\mathcal{T}^2,\bar z_2)_{\alpha_2}{}^{A_2} \\
&\quad = \sum_{\substack{\mathcal{T}^k,A_k,\alpha_k\\ \beta_1,\beta_2}} \bar z_{12}^{\Delta^g(\mathcal{T}^k)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)} \Big\{ \Big[1+\frac{1}{2}\sum_{r=1}^{\infty}\frac{\bar z_{12}^r}{r!}\bar\partial_2^r\Big] g_-(\mathcal{T}^k,\bar z_2)_{\alpha_k}{}^{A_k}\delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2} \\
&\quad - \sum_{r=0}^{\infty} \frac{\bar z_{12}^{r+1}}{(r+1)!} \bar\partial_2^r \{2L_g^{ab} : g_-(\mathcal{T}^k,\bar z_2)_{\alpha_k}{}^{A_k}\bar J_b(\bar z_2) :\}\frac{1}{2}[(\mathcal{T}_a^1)_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}^{\beta_2} - (\mathcal{T}_a^2)_{\alpha_2}{}^{\beta_2}\delta_{\alpha_1}^{\beta_1}]\Big\} \\
&\qquad\qquad \times \bar F_{\beta_1\beta_2 A_k}^{A_1A_2\alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) + \mathcal{O}(k^{-1}) \quad .
\end{aligned}
\tag{5.12}
$$

is also obtained with the high-level ODE (4.16).

<u>PDE's for operator products</u>

Using the algebra (2.19a,b) and the ODE's (2.21a,b) for $g_\pm$, a set of exact PDE's for the product of two vertex operators are easily derived [26]:

$$P_+(z_1,z_2)_{A_1A_2}{}^{\alpha_1\alpha_2} \equiv g_+(\mathcal{T}^1,z_1)_{A_1}{}^{\alpha_1} g_+(\mathcal{T}^2,z_2)_{A_2}{}^{\alpha_2} \tag{5.13a}$$

$$\partial_1 P_+(z_1,z_2) = 2L_g^{ab}\{: P_+(z_1,z_2)J_a(z_1) : \mathcal{T}_b^1 + \frac{P_+(z_1,z_2)\mathcal{T}_a^1\mathcal{T}_b^2}{z_{12}}\} \tag{5.13b}$$

$$\partial_2 P_+(z_1,z_2) = 2L_g^{ab}\{: P_+(z_1,z_2)J_a(z_2) : \mathcal{T}_b^2 - \frac{P_+(z_1,z_2)\mathcal{T}_a^1\mathcal{T}_b^2}{z_{12}}\} \tag{5.13c}$$

$$P_-(\bar z_1,\bar z_2)_{\alpha_1\alpha_2}{}^{A_1A_2} \equiv g_-(\mathcal{T}^1,\bar z_1)_{\alpha_1}{}^{A_1} g_-(\mathcal{T}^2,\bar z_2)_{\alpha_2}{}^{A_2} \tag{5.13d}$$

$$\bar\partial_1 P_-(\bar z_1,\bar z_2) = -2L_g^{ab}\{\mathcal{T}_b^1 : P_-(\bar z_1,\bar z_2)\bar J_a(\bar z_1) : - \frac{\mathcal{T}_a^1\mathcal{T}_b^2 P_-(\bar z_1,\bar z_2)}{\bar z_{12}}\} \tag{5.13e}$$

$$\bar\partial_2 P_-(\bar z_1,\bar z_2) = -2L_g^{ab}\{\mathcal{T}_b^2 : P_-(\bar z_1,\bar z_2)\bar J_a(\bar z_2) : + \frac{\mathcal{T}_a^1\mathcal{T}_b^2 P_-(\bar z_1,\bar z_2)}{\bar z_{12}}\} \quad . \tag{5.13f}$$

The normal ordering here is the same as in (2.10) with $g \to P$, and we have checked explicitly that the OPE's (5.6) and (5.11) satisfy these PDE's thru the appropriate order.

Braid relations

We close this section with a short discussion of braid relations which provides a check of our formulation against standard relations in the literature.

The braid matrix (or universal R matrix) $\mathcal{B}_g$, which acts on the quantum group indices of the chiral vertex operator is defined by the braid relation

$$g_+(\mathcal{T}^1,z_1)_{A_1}{}^{\alpha_1} g_+(\mathcal{T}^2,z_2)_{A_2}{}^{\alpha_2} = \mathcal{B}_g(\mathcal{T}^1\mathcal{T}^2)_{A_1A_2}{}^{B_2B_1} g_+(\mathcal{T}^2,z_2)_{B_2}{}^{\alpha_2} g_+(\mathcal{T}^1,z_1)_{B_1}{}^{\alpha_1} \tag{5.14a}$$

$$\mathcal{B}_g(\mathcal{T}^2\mathcal{T}^1) = \mathcal{B}_g^{-1}(\mathcal{T}^1\mathcal{T}^2) \quad . \tag{5.14b}$$

Using (5.14) and (3.18) in the OPE (5.6), we find that the braid matrix also describes the $1 \leftrightarrow 2$ exchange of the fusion coefficient F of the zero modes in (5.3),

$$\begin{aligned} &F^{\alpha_1\alpha_2A_3}_{A_1A_2\alpha_3}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^3) \\ &= \mathcal{B}_g(\mathcal{T}^1\mathcal{T}^2)_{A_1A_2}{}^{B_2B_1} F^{\alpha_2\alpha_1A_3}_{B_2B_1\alpha_3}(\mathcal{T}^2\mathcal{T}^1\mathcal{T}^3) \mathrm{e}^{-i\pi[\Delta^g(\mathcal{T}^3)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)]\mathrm{sign}(\arg(z_1/z_2))} + \mathcal{O}(k^{-3/2}) \,. \end{aligned} \tag{5.15}$$

Using next the factorized form of F in (5.4) and the exchange relation of the Clebsch-Gordan coefficient C,

$$C^{\alpha_2\alpha_1}{}_{\alpha_3}(\mathcal{T}^2\mathcal{T}^1\mathcal{T}^3) = \nu(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^3) C^{\alpha_1\alpha_2}{}_{\alpha_3}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^3) \tag{5.16a}$$

$$\nu(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^3) = \nu(\mathcal{T}^2\mathcal{T}^1\mathcal{T}^3) = \pm 1 \tag{5.16b}$$

we see that the braid matrix also describes the $1 \leftrightarrow 2$ exchange of the quantum Clebsch-Gordan coefficients Q

$$\begin{aligned} Q_{A_1A_2}{}^{A_3}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^3) =& \mathcal{B}_g(\mathcal{T}^1\mathcal{T}^2)_{A_1A_2}{}^{B_2B_1} Q_{B_2B_1}{}^{A_3}(\mathcal{T}^2\mathcal{T}^1\mathcal{T}^3) \\ &\times \nu(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^3) \mathrm{e}^{-i\pi[\Delta^g(\mathcal{T}^3)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)]\mathrm{sign}(\arg(z_1/z_2))} + \mathcal{O}(k^{-3/2}) \quad . \end{aligned} \tag{5.17}$$

To discuss this relation further, we introduce the invariant level $x = 2k/\psi_g^2$ of the affine algebra, where ψ_g is the highest root of g. When $\arg(z_2/z_1) > 0$ and the quantum group parameter q is identified as usual as

$$q = \mathrm{e}^{\frac{2\pi i}{x}} + \mathcal{O}(k^{-3/2}) \tag{5.18}$$

we find that the relation (5.17) agrees with the high-level form of the corresponding quantum $SU(2)$ relation

$$K^{j_1j_2}_j R^{j_2j_1} = (-1)^{j_1+j_2-j} q^{(c_j-c_{j_1}-c_{j_2})/2} K^{j_2j_1}_j \quad , \quad c_j = j(j+1) \quad , \quad q = \exp\left(\frac{2\pi i}{x+2}\right) \quad . \tag{5.19}$$

which appears as eq.(13) in Ref.[30]. To see this agreement, use the $SU(2)$ identifications

$$\mathcal{B}_g(\mathcal{T}^1\mathcal{T}^2) \to R^{j_1 j_2} \quad , \quad \Delta_g(\mathcal{T}) \to \frac{j(j+1)}{x+2} \tag{5.20a}$$

$$Q(T^1T^2T^3) \to K_j^{j_1 j_2} \quad , \quad \nu(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^3) \to (-1)^{j_1+j_2-j} \tag{5.20b}$$

and the inverse relation (5.14b) to move $\mathcal{B}_g$ to the left of (5.17).

6 Chiral Correlators

Using the semiclassical vertex operators (4.9) and the algebra (4.7), one straightforwardly computes the semiclassical chiral correlators (2.22b),

$$A_g^+(\mathcal{T},z) = \langle 0|G_+(\mathcal{T}^1)\cdots G_+(\mathcal{T}^n)|0\rangle \left[1 + 2L_g^{ab}\sum_{i<j}^{n} \mathcal{T}_a^i \mathcal{T}_b^j \ln z_{ij}\right] + \mathcal{O}(k^{-3/2}) \tag{6.1}$$

up to the constant zero-mode averages $\langle G_+ \cdots G_+\rangle$. This simple result is obtained because the normal-ordered terms in (4.9) once again fail to contribute to the correlators, and the result (6.1) indeed satisfies the chiral KZ equations (2.14a). Using the algebra (4.7a), one finds that the zero-mode averages, and hence the chiral correlators (6.1), satisfy the g-global Ward identities

$$\langle 0|G_+(\mathcal{T}^1)\cdots G_+(\mathcal{T}^n)|0\rangle \sum_{i=1}^{n} \mathcal{T}_a^i = 0 + \mathcal{O}(k^{-3/2}) \tag{6.2a}$$

$$A_g^+(\mathcal{T},z)\sum_{i=1}^{n} \mathcal{T}_a^i = 0 + \mathcal{O}(k^{-3/2}) \tag{6.2b}$$

so that these quantities are g-invariant thru this order of the semiclassical expansion. As discussed in Ref.[25], the chiral correlators, and hence the zero-mode averages, are similarly invariant under the quantum group.

To be more explicit about the zero-mode averages, we may use the fusion relation (5.3) to obtain the forms,

$$\langle 0|G_+(\mathcal{T})_A{}^{\alpha}|0\rangle = \delta(\mathcal{T},\mathcal{T}_{(1)}) \tag{6.3a}$$

$$\langle 0|G_+(\mathcal{T}^1)_{A_1}{}^{\alpha_1} G_+(\mathcal{T}^2)_{A_2}{}^{\alpha_2}|0\rangle = \delta(\mathcal{T}^2,\bar{\mathcal{T}}^1) F_{A_1 A_2 \cdot}^{\alpha_1\alpha_2 \cdot}(\mathcal{T}^1\bar{\mathcal{T}}^1\mathcal{T}_{(1)}) + \mathcal{O}(k^{-3/2}) \tag{6.3b}$$

$$\begin{aligned}&\langle 0|G_+(\mathcal{T}^1)_{A_1}{}^{\alpha_1} G_+(\mathcal{T}^2)_{A_2}{}^{\alpha_2} G_+(\mathcal{T}^2)_{A_3}{}^{\alpha_3}|0\rangle \\ &\quad = F_{A_1A_2\bar{\alpha}_3}^{\alpha_1\alpha_2\bar{A}_3}(\mathcal{T}^1\mathcal{T}^2\bar{\mathcal{T}}^3) F_{\bar{A}_3A_3\cdot}^{\bar{\alpha}_3\alpha_3\cdot}(\bar{\mathcal{T}}^3\mathcal{T}^3\mathcal{T}_{(1)}) + \mathcal{O}(k^{-3/2})\end{aligned} \tag{6.3c}$$

$$\begin{aligned}&\langle 0|G_+(\mathcal{T}^1)_{A_1}{}^{\alpha_1} G_+(\mathcal{T}^2)_{A_2}{}^{\alpha_2} G_+(\mathcal{T}^3)_{A_3}{}^{\alpha_3} G_+(\mathcal{T}^4)_{A_4}{}^{\alpha_4}|0\rangle \\ &\quad = \sum_{\mathcal{T}^k \, {A_k,\bar{A}_k \atop \alpha_k,\bar{\alpha}_k}} F_{A_1A_2\alpha_k}^{\alpha_1\alpha_2A_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) F_{A_3A_4\bar{\alpha}_k}^{\alpha_3\alpha_4\bar{A}_k}(\mathcal{T}^3\mathcal{T}^4\bar{\mathcal{T}}^k) F_{A_k\bar{A}_k\cdot}^{\alpha_k\bar{\alpha}_k\cdot}(\mathcal{T}^k\bar{\mathcal{T}}^k\mathcal{T}_{(1)}) + \mathcal{O}(k^{-3/2})\end{aligned} \tag{6.3d}$$

where δ is Kronecker delta and $\mathcal{T}_{(1)}$ is the trivial representation. Using (5.4), the zero-mode averages can also be expressed in terms of (level-independent) classical and (level-dependent) quantum group invariants $\bar{v}$ and d

$$\langle 0|G_+(\mathcal{T}^1)\cdots G_+(\mathcal{T}^n)|0\rangle_A{}^\alpha = \sum_m d_A^m \bar{v}_m^\alpha + \mathcal{O}(k^{-3/2}) \tag{6.4a}$$

$$v_\alpha^m \bar{v}_m^\beta = (I_g^n)_\alpha{}^\beta \quad , \quad \bar{v}_m^\alpha v_\alpha^l = \delta_m^l \tag{6.4b}$$

$$d_A^m \bar{d}_m^B = (D_g^n)_A{}^B \quad , \quad \bar{d}_m^A d_A^l = \delta_m^l \tag{6.4c}$$

where we have also introduced the dual invariants v and $\bar{d}$ to write the completeness and orthonormality relations (6.4b,c) of the invariants. The quantity D_g^n is the quantum analogue of the classical invariant projector I_g^n.

In further detail, we find for the four-point average

$$\langle 0|G_+(\mathcal{T}^1)_{A_1}{}^{\alpha_1} G_+(\mathcal{T}^2)_{A_2}{}^{\alpha_2} G_+(\mathcal{T}^3)_{A_3}{}^{\alpha_3} G_+(\mathcal{T}^4)_{A_4}{}^{\alpha_4}|0\rangle = \sum_m d(\mathrm{s},g)_A^m \bar{v}(\mathrm{s},g)_m^\alpha + \mathcal{O}(k^{-3/2}) \tag{6.5a}$$

$$\bar{v}(s,g)_m^\alpha \equiv \frac{1}{\sqrt{\dim \mathcal{T}^m}} \sum_{\alpha_m,\bar{\alpha}_m} C^{\alpha_1\alpha_2}{}_{\alpha_m}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^m) C^{\alpha_3\alpha_4}{}_{\bar{\alpha}_m}(\mathcal{T}^3\mathcal{T}^4\bar{\mathcal{T}}^m)\eta^{\alpha_m\bar{\alpha}_m}(\mathcal{T}^m) \tag{6.5b}$$

$$d(s,g)_A^m \equiv \frac{1}{\sqrt{\mathrm{Tr}\,\Lambda(\mathcal{T}^m)}} \sum_{A_m,\bar{A}_m} Q_{A_1A_2}{}^{A_m}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^m) Q_{A_3A_4}{}^{\bar{A}_m}(\mathcal{T}^3\mathcal{T}^4\bar{\mathcal{T}}^m)\Lambda_{A_m\bar{A}_m}(\mathcal{T}^m) \tag{6.5c}$$

where C and Q are the classical and quantum Clebsch-Gordan coefficients, and $\eta_{\alpha\beta}(\mathcal{T})$ and $\Lambda_{AB}(\mathcal{T})$ are defined in (5.10h) and (5.10i). The form (6.5b) of $\bar{v}(s,g)_m$ was first given in Ref.[42], where these quantities are called the s-channel invariants of the four-point correlator. Similarly, the quantities $d(s,g)^m$ may be interpreted as the s-channel quantum invariants of the 4-point correlator.

The classical and quantum invariants $\bar{v}$ and d also appear in the conformal-block expansion of the chiral correlators. Using the KZ gauge

$$A_g^+(z_1,z_2,z_3,z_4)_A{}^\alpha = \frac{Y_g^+(y)_A{}^\alpha}{\prod_{i<j}^4 z_{ij}^{\gamma_{ij}^g}} \quad , \quad y = \frac{z_{12}z_{34}}{z_{14}z_{32}} \tag{6.6a}$$

$$\gamma_{12}^g = \gamma_{13}^g = 0 \quad , \quad \gamma_{14}^g = 2\Delta^g(\mathcal{T}^1) \quad , \quad \gamma_{23}^g = \Delta^g(\mathcal{T}^1) + \Delta^g(\mathcal{T}^2) + \Delta^g(\mathcal{T}^3) - \Delta^g(\mathcal{T}^4) \tag{6.6b}$$

$$\gamma_{24}^g = -\Delta^g(\mathcal{T}^1)+\Delta^g(\mathcal{T}^2)-\Delta^g(\mathcal{T}^3)+\Delta^g(\mathcal{T}^4) \ , \ \gamma_{34}^g = -\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)+\Delta^g(\mathcal{T}^3)+\Delta^g(\mathcal{T}^4) \tag{6.6c}$$

and the g-global Ward identity in (2.13), we find from (6.1) and (6.5) that

$$Y_g^+(y)_A{}^\alpha = \sum_{m,l} d(\mathrm{s},g)_A^m \mathcal{F}_g^{(\mathrm{s})}(y)_m{}^l \bar{v}(\mathrm{s},g)_l^\alpha + \mathcal{O}(k^{-3/2}) \tag{6.7a}$$

$$\mathcal{F}_g^{(\mathrm{s})}(y)_m{}^l = \bar{v}(\mathrm{s},g)_m^\alpha[\mathbb{1} + 2L_g^{ab}(\mathcal{T}_a^1\mathcal{T}_b^2 \ln y + \mathcal{T}_a^1\mathcal{T}_b^3\ln(1-y))]_\alpha{}^\beta v(\mathrm{s},g)_\beta^l + \mathcal{O}(k^{-3/2}) \tag{6.7b}$$

where $\mathcal{F}_g^{(s)}$ are the semiclassical s-channel affine-Sugawara blocks obtained in Ref.[42]. The reproduction of the affine-Sugawara blocks in (6.7) is a central check on the new nonabelian chiral vertex operators (4.2). We emphasize however that, in the conventional block analysis [7,42] of the solutions of the KZ equations, the quantum invariants d_A^m in (6.7a) are discarded as irrelevant constants.

The corresponding results for the antichiral correlators (2.22c) are,

$$A_g^-(\mathcal{T},\bar{z}) = \left[1 + 2L_g^{ab}\sum_{i<j}^{n} \mathcal{T}_a^i \mathcal{T}_b^j \ln \bar{z}_{ij}\right] \langle 0|G_-(\mathcal{T}^1)\cdots G_-(\mathcal{T}^n)|0\rangle + \mathcal{O}(k^{-3/2}) \tag{6.8a}$$

$$\langle 0|G_-(\mathcal{T}^1)\cdots G_-(\mathcal{T}^n)|0\rangle_\alpha{}^A = \sum_m v_\alpha^m \bar{d}_m^A + \mathcal{O}(k^{-3/2}) \tag{6.8b}$$

$$A_g^-(\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4)_\alpha{}^A = \frac{Y_g^-(\bar{y})_\alpha{}^A}{\prod_{i<j}^4 \bar{z}_{ij}^{\gamma_{ij}^g}} \tag{6.8c}$$

$$Y_g^-(\bar{y})_\alpha{}^A = \sum_{m,l} v(\mathrm{s},g)_\alpha^m \mathcal{F}_g^{(\mathrm{s})}(\bar{y})_m{}^l \bar{d}(\mathrm{s},g)_l^A + \mathcal{O}(k^{-3/2}) \tag{6.8d}$$

$$\mathcal{F}_g^{(\mathrm{s})}(\bar{y})_m{}^l = (\mathcal{F}_g^{(\mathrm{s})}(y)_l{}^m)^* \tag{6.8e}$$

where v and $\bar{d}$ are the dual invariants to $\bar{v}$ and d.

Braiding and crossing

The braid matrix $\mathcal{B}_g$ in [25]

$$g_+(\mathcal{T}^2,z_2)_{A_2}{}^{\alpha_2} g_+(\mathcal{T}^3,z_3)_{A_3}{}^{\alpha_3} = \mathcal{B}_g(\mathcal{T}^2\mathcal{T}^3)_{A_2A_3}{}^{B_3B_2} g_+(\mathcal{T}^3,z_3)_{B_3}{}^{\alpha_3} g_+(\mathcal{T}^2,z_2)_{B_2}{}^{\alpha_2} \tag{6.9}$$

describes the exchange of two chiral vertex operators, while the s-u crossing matrix $X_g(\mathrm{su})$ in Ref.[42]

$$\mathcal{F}_g^{(\mathrm{s})}(y)_m{}^l = X_g(\mathrm{su})_m{}^p \mathcal{F}_g^{(\mathrm{u})}(y)_p{}^q X_g^{-1}(\mathrm{su})_q{}^l + \mathcal{O}(k^{-2}) \tag{6.10a}$$

$$\bar{v}(\mathrm{s},g)_m = X_g(\mathrm{su})_m{}^l \bar{v}(\mathrm{u},g)^l + \mathcal{O}(k^{-2}) \tag{6.10b}$$

$$X_g(\mathrm{su})_m{}^l = \bar{v}(\mathrm{s},g)_m v(\mathrm{u},g)^l + \mathcal{O}(k^{-2}) \tag{6.10c}$$

relates the s and u channel conformal blocks $\mathcal{F}_g^{(\mathrm{s})}$ and $\mathcal{F}_g^{(\mathrm{u})}$ of the affine-Sugawara construction. It is clear that $\mathcal{B}_g$ and X_g represent the same physical operation, namely the exchange of external states $2 \leftrightarrow 3$. The braid matrix acts however in the quantum space with $A_i, B_i = 1\ldots\dim\mathcal{T}^i$, while the crossing matrix acts on the generally smaller space of conformal blocks $\mathcal{F}_g$ with $m,l = 1\ldots\dim(\text{invariants})$. It follows that there must be a map, or intertwining relation, between the braid matrix $(\mathcal{B}_g)_A{}^B$ and the crossing matrix $(X_g)_m{}^l$ and we shall find that the intertwiner is the set of quantum invariants d_A^m.

To see this, we first translate the braid relation (6.9) into the braid relation of the chiral correlator A_g^+

$$A_g^+(z_1,z_2,z_3,z_4)_{A_1A_2A_3A_4}^{\alpha_1\alpha_2\alpha_3\alpha_4} = \mathcal{B}_g(\mathcal{T}^2\mathcal{T}^3)_{A_2A_3}{}^{B_3B_2} A_g^+(z_1,z_3,z_2,z_4)_{A_1B_3B_2A_4}^{\alpha_1\alpha_3\alpha_2\alpha_4} \quad . \tag{6.11}$$

In terms of the invariant four-point correlator Y_g^+, this relation reads

$$
\begin{aligned}
&Y_g^+(y)^{\alpha_1\alpha_2\alpha_3\alpha_4}_{A_1A_2A_3A_4} \\
&=\mathcal{B}_g(\mathcal{T}^2\mathcal{T}^3)_{A_2A_3}{}^{B_3B_2} Y_g^+(1-y)^{\alpha_1\alpha_3\alpha_2\alpha_4}_{A_1B_3B_2A_4} \mathrm{e}^{i\pi(\Delta^g(\mathcal{T}^1)+\Delta^g(\mathcal{T}^2)+\Delta^g(\mathcal{T}^3)-\Delta^g(\mathcal{T}^4))\mathrm{sign}(\arg(z_2/z_3))} \\
&\qquad\qquad + \mathcal{O}(k^{-3/2}) \quad .
\end{aligned}
\tag{6.12}
$$

Then using the block decomposition (6.7a) along with the similar u-channel decomposition

$$
Y_g^+(1-y)_{A_1A_3A_2A_4}{}^{\alpha_1\alpha_3\alpha_2\alpha_4} = \sum_{m,l} d(\mathrm{s},g)^m_{A_1A_3A_2A_4} \mathcal{F}_g^{(\mathrm{u})}(y)_m{}^l \bar{v}(\mathrm{u},g)_l^{\alpha_1\alpha_2\alpha_3\alpha_4} + \mathcal{O}(k^{-3/2}) \tag{6.13}
$$

we obtain the intertwining relation

$$
\begin{aligned}
&\mathcal{B}_g(\mathcal{T}^2\mathcal{T}^3)_{A_2A_3}{}^{B_3B_2} d(\mathrm{s},g)^m_{A_1B_3B_2A_4} \\
&=\sum_l d(\mathrm{s},g)^l_{A_1A_3A_2A_4} X_g(su)_l{}^m \mathrm{e}^{-i\pi[\Delta^g(\mathcal{T}^1)+\Delta^g(\mathcal{T}^2)+\Delta^g(\mathcal{T}^3)-\Delta^g(\mathcal{T}^4)]\mathrm{sign}(\arg(z_2/z_3))} + \mathcal{O}(k^{-3/2}) \quad .
\end{aligned}
\tag{6.14}
$$

Here, we have also used the crossing relations (6.10) and completeness of the g-invariants $\bar{v}(\mathrm{s},g)$. For the special case of certain irreps of $SU(n)$, this relation was obtained exactly in Ref.[40].

7 Chiral-Antichiral OPE's

We turn next to the question of the chiral-antichiral OPE's, which are known from the action formulation [34,35,37,40,41] to depend on the treatment of the constant quantum space ambiguity in the factorization $g(\mathcal{T},\bar{z},z)_\alpha{}^\beta = g_-(\mathcal{T},\bar{z})_\alpha{}^A g_+(\mathcal{T},z)_A{}^\beta$. In what follows we briefly discuss this ambiguity, as it is reflected in our operator formulation.

We begin by assuming the exact operator solution

$$
[J_a(m), g_-(\mathcal{T},\bar{z})] = [\bar{J}_a(m), g_+(\mathcal{T},z)] = 0 \tag{7.1}
$$

of the conditions (2.20). The choice (7.1) is the natural solution of (2.20) because we now know that g_+ and g_- are unitary (and hence invertible) in the extreme semiclassical limit. Moreover, (7.1) immediately implies the natural relations

$$
[\bar{T}_g(\bar{z}), g_+(\mathcal{T},z)] = [T_g(z), g_-(\mathcal{T},\bar{z})] = 0 \tag{7.2}
$$

which say that the chiral and antichiral vertex operators are inert under the antichiral and chiral affine-Sugawara constructions respectively.

Using (7.1), one can also derive the regular PDE's

$$
P_{+-}(z_1,\bar{z}_2)_{A_1\alpha_2}{}^{\alpha_1A_2} \equiv g_+(\mathcal{T}^1,z_1)_{A_1}{}^{\alpha_1} g_-(\mathcal{T}^2,\bar{z}_2)_{\alpha_2}{}^{A_2} \tag{7.3a}
$$

$$\partial_1 P_{+-}(z_1,\bar z_2) = 2L_g^{ab} : P_{+-}(z_1,\bar z_2) E_a(z_1) : \mathcal{T}_b^1 \tag{7.3b}$$

$$\bar\partial_2 P_{+-}(z_1,\bar z_2) = -2L_g^{ab} : P_{+-}(z_1,\bar z_2) \bar J_a(\bar z_2) : \mathcal{T}_b^2 \tag{7.3c}$$

$$P_{-+}(\bar z_2, z_1)_{\alpha_2 A_1}{}^{A_2\alpha_1} \equiv g_-(\mathcal{T}^2,\bar z_2)_{\alpha_2}{}^{A_2} g_+(\mathcal{T}^1, z_1)_{A_1}{}^{\alpha_1} \tag{7.3d}$$

$$\partial_1 P_{-+}(\bar z_2, z_1) = 2L_g^{ab} : P_{-+}(\bar z_2, z_1) E_a(z_1) : \mathcal{T}_b^1 \tag{7.3e}$$

$$\bar\partial_2 P_{-+}(\bar z_2, z_1) = -2L_g^{ab} : P_{-+}(\bar z_2, z_1) \bar J_a(\bar z_2) : \mathcal{T}_b^2 \tag{7.3f}$$

for the chiral-antichiral products, and these differential equations tell us immediately that the chiral-antichiral operator products and OPE's are regular,

$$g_+(\mathcal{T}^1, z_1) g_-(\mathcal{T}^2,\bar z_2) = \text{regular in } (z_1 - \bar z_2) \tag{7.4a}$$

$$g_-(\mathcal{T}^2,\bar z_2) g_+(\mathcal{T}^1, z_1) = \text{regular in } (\bar z_2 - z_1) \quad . \tag{7.4b}$$

Because of possibly non-trivial braiding however, the regularity of these OPE's does not necessarily imply that the chiral and antichiral vertex operators commute.

In our semiclassical expansion, one can be more explicit about the chiral-antichiral operator products. We first need the chiral-antichiral commutators of the currents with the zero modes

$$[J_a(0), G_-(\mathcal{T})] = [\bar J_a(0), G_+(\mathcal{T}] = 0 + \mathcal{O}(k^{-3/2}) \tag{7.5a}$$

$$[J_a(m\neq 0), G_-(\mathcal{T})] = [\bar J_a(m\neq 0), G_+(\mathcal{T}] = 0 + \mathcal{O}(k^{-1}) \tag{7.5b}$$

which follow from (7.1) and the expressions (4.11), (4.22) for the zero modes in terms of the primary fields. The results (4.7), (4.20) and (7.5) collect the complete semiclassical algebra of the currents with the chiral and antichiral zero modes.

The commutators (7.5) are the natural solutions (because $G_+(\mathcal{T})$ and $G_-(\mathcal{T})$ are also unitary in the extreme semiclassical limit) of the general chiral-antichiral conditions

$$[J_a(0), G_-(\mathcal{T})] G_+(\mathcal{T}) = 0 + \mathcal{O}(k^{-3/2}) \quad , \quad [J_a(m\neq 0), G_-(\mathcal{T})] G_+(\mathcal{T}) = 0 + \mathcal{O}(k^{-1}) \tag{7.6a}$$

$$G_-(\mathcal{T})[\bar J_a(0), G_+(\mathcal{T})] = 0 + \mathcal{O}(k^{-3/2}) \quad , \quad G_-(\mathcal{T})[\bar J_a(m\neq 0), G_+(\mathcal{T})] = 0 + \mathcal{O}(k^{-1}) \tag{7.6b}$$

which are the zero-mode analogues of the general conditions (2.20). The relations (7.6a,b) follow directly from (2.20), without the assumption (7.1), using only the current algebra and the inversions (4.11) and (4.22).

Continuing with (7.5), and following the steps of Section 5, one obtains the chiral-antichiral operator products as follows,

$$g_+(\mathcal{T}^1, z_1)_{A_1}{}^{\alpha_1} g_-(\mathcal{T}^2,\bar z_2)_{\alpha_2}{}^{A_2} = G_+(\mathcal{T}^1)_{A_1}{}^{\beta_1} G_-(\mathcal{T}^2)_{\beta_2}{}^{A_2} M(z_1,\bar z_2)_{\beta_1\alpha_2}{}^{\alpha_1\beta_2} + \mathcal{O}(k^{-3/2}) \tag{7.7a}$$

$$g_-(\mathcal{T}^2,\bar z_2)_{\alpha_2}{}^{A_2} g_+(\mathcal{T}^1, z_1)_{A_1}{}^{\alpha_1} = G_-(\mathcal{T}^2)_{\beta_2}{}^{A_2} G_+(\mathcal{T}^1)_{A_1}{}^{\beta_1} M(z_1,\bar z_2)_{\beta_1\alpha_2}{}^{\alpha_1\beta_2} + \mathcal{O}(k^{-3/2}) \tag{7.7b}$$

$$[g_+(\mathcal{T}^1, z_1), g_-(\mathcal{T}^2,\bar z_2)] = [G_+(\mathcal{T}^1), G_-(\mathcal{T}^2)] M(z_1,\bar z_2) + \mathcal{O}(k^{-3/2}) \tag{7.7c}$$

where the explicit form of $M(z_1, \bar{z}_2)$ is

$$\begin{aligned} M(z_1, \bar{z}_2) = \mathbb{1} + iX^a(z_1)\mathcal{T}_a^1 - i\bar{X}^a(\bar{z}_2)\mathcal{T}_a^2 + X^a(z_1)\bar{X}^b(\bar{z}_2)\mathcal{T}_a^1\mathcal{T}_b^2 \\ + N^{ab}(z_1)\mathcal{T}_b^1\mathcal{T}_a^1 + \bar{N}^{ab}(\bar{z}_2)\mathcal{T}_a^2\mathcal{T}_b^2 + \mathcal{O}(k^{-3/2}) \quad . \end{aligned} \tag{7.8}$$

The form (7.8) for $M(z_1, \bar{z}_2)$ can easily be expanded to find the explicit semiclassical forms of the regular OPE's (7.4).

The result (7.7) shows that the ambiguity of the chiral-antichiral OPE's, and hence the chiral-antichiral commutator, is entirely in the constant zero-mode products $G_+(\mathcal{T})G_-(\mathcal{T})$ and $G_-(\mathcal{T})G_+(\mathcal{T})$, which then carry the constant quantum space ambiguity [35,40,41] of the factorization. Among the various possibilities however, the most esthetic solution is the one in which the chiral and antichiral sectors commute

$$[G_+(\mathcal{T}^1), G_-(\mathcal{T}^2)] = 0 + \mathcal{O}(k^{-3/2}) \tag{7.9a}$$

$$\Rightarrow [g_+(\mathcal{T}^1, z), g_-(\mathcal{T}^2, \bar{z})] = 0 + \mathcal{O}(k^{-3/2}) \tag{7.9b}$$

which is the "gauge choice" discussed by Caneschi and Lysiansky in Ref.[41]. In this case, we also obtain the algebra

$$[G_+(\mathcal{T}^1), g_-(\mathcal{T}^2, \bar{z})] = [g_+(\mathcal{T}^1, z), G_-(\mathcal{T}^2)] = 0 + \mathcal{O}(k^{-3/2}) \tag{7.10}$$

of the chiral zero modes with the antichiral vertex operators and vice versa. Unless otherwise stated, we limit the discussion below to the gauge choice (7.9).

8 Semiclassical WZW Vertex Operators

Our final task is to assemble the semiclassical chiral and antichiral sectors into semiclassical WZW theory, beginning with the semiclassical WZW vertex operators. Using (2.18), (4.9) and (4.21c) we find

$$g(\mathcal{T}, \bar{z}, z)_\alpha{}^\beta = g_-(\mathcal{T}, \bar{z})_\alpha{}^A g_+(\mathcal{T}, z)_A{}^\beta \tag{8.1a}$$

$$= \bar{z}^{-2\Delta^g(\mathcal{T})}[\mathbb{1} - i\mathcal{T}_a\bar{X}^a(\bar{z}) + \mathcal{T}_a\mathcal{T}_b\bar{N}^{ab}(\bar{z})]_\alpha{}^\rho G(\mathcal{T})_\rho{}^\sigma[\mathbb{1} + iX^a(z)\mathcal{T}_a + N^{ab}(z)\mathcal{T}_b\mathcal{T}_a]_\sigma{}^\beta + \mathcal{O}(k^{-3/2}) \tag{8.1b}$$

$$G(\mathcal{T})_\alpha{}^\beta \equiv \sum_A G_-(\mathcal{T})_\alpha{}^A G_+(\mathcal{T})_A{}^\beta \tag{8.1c}$$

$$G(\mathcal{T}) = \mathcal{O}(k^0) \tag{8.1d}$$

$$\partial G(\mathcal{T}) = \bar{\partial} G(\mathcal{T}) = 0 + \mathcal{O}(k^{-3/2}) \tag{8.1e}$$

where $G(\mathcal{T})$ in (8.1c) is the WZW zero mode.

Here are some important properties of the WZW zero mode.

A. Semiclassical unitarity. The extreme semiclassical unitarity of $G(\mathcal{T})$,

$$G^\dagger(\mathcal{T})G(\mathcal{T}) = 1\!\!1 + \mathcal{O}(k^{-1}) \tag{8.2}$$

follows from the extreme semiclassical unitarity of $G_+(\mathcal{T})$ and $G_-(\mathcal{T})$. This property is independent of the gauge choice (7.9).

B. Algebra with the currents. The algebra of the currents with the WZW zero mode

$$[J_a(0), G(\mathcal{T})] = G(\mathcal{T})\mathcal{T}_a + \mathcal{O}(k^{-3/2}) \tag{8.3a}$$

$$[J_a(m \neq 0), G(\mathcal{T})] = \frac{i}{2km} : G(\mathcal{T})J_b(m) : f_a{}^{bc}\mathcal{T}_c + \mathcal{O}(k^{-1}) \tag{8.3b}$$

$$[\bar{J}_a(0), G(\mathcal{T})] = -\mathcal{T}_a G(\mathcal{T}) + \mathcal{O}(k^{-3/2}) \tag{8.3c}$$

$$[\bar{J}_a(m \neq 0), G(\mathcal{T})] = -\frac{i}{2km} f_a{}^{bc}\mathcal{T}_c : G(\mathcal{T})\bar{J}_b(m) : + \mathcal{O}(k^{-1}) \tag{8.3d}$$

follows easily from (4.7), (4.20) and the gauge choice (7.9a). Curiously, these relations can also be derived from (4.7), (4.20) and the general conditions (7.6), so they are in fact also independent of the gauge choice (7.9). The algebra (8.3) is consistent with the semiclassical unitarity of $G(\mathcal{T})$ in (8.2).

C. Fusion and group multiplication. The fusion rule for two WZW zero modes,

$$G(\mathcal{T}^1)_{\alpha_1}{}^{\beta_1} G(\mathcal{T}^2)_{\alpha_2}{}^{\beta_2} = \sum_{\mathcal{T}_k,\alpha_k,\beta_k} D^{\beta_1\beta_2\alpha_k}_{\alpha_1\alpha_2\beta_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) G(\mathcal{T}^k)_{\alpha_k}{}^{\beta_k} + \mathcal{O}(k^{-3/2}) \tag{8.4a}$$

$$D^{\beta_1\beta_2\alpha_k}_{\alpha_1\alpha_2\beta_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) = \bar{C}_{\alpha_1\alpha_2}{}^{\alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) C^{\beta_1\beta_2}{}_{\beta_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) \tag{8.4b}$$

follows from (7.9a), (5.3), (5.4), (5.10e,f) and (5.10l), where C and $\bar{C}$ are the classical Clebsch-Gordan coefficients and their duals defined in (5.10j). To this order of the semiclassical expansion, the fusion rule (8.4) is the same as the multiplication law for classical group elements in the Clebsch basis.

D. Averages. Using (7.9a), (6.4) and (6.8b), we find that the zero-mode averages are Haar integrals

$$\begin{aligned}
\langle 0|G(\mathcal{T}^1)\cdots G(\mathcal{T}^n)|0\rangle_\alpha{}^\beta &= \langle 0|G_-(\mathcal{T}^1)\cdots G_-(\mathcal{T}^n)|0\rangle_\alpha{}^A \langle 0|G_+(\mathcal{T}^1)\cdots G_+(\mathcal{T}^n)|0\rangle_A{}^\beta \\
&= \sum_{m,l,A} v_\alpha^m \bar{d}_m^A d_A^l \bar{v}_l^\beta + \mathcal{O}(k^{-3/2}) = \sum_m v_\alpha^m \bar{v}_m^\beta + \mathcal{O}(k^{-3/2}) \\
&= I_g^n + \mathcal{O}(k^{-3/2}) = \int \mathrm{d}\mathcal{G}\, \mathcal{G}(\mathcal{T}^1)\cdots\mathcal{G}(\mathcal{T}^n) + \mathcal{O}(k^{-3/2})
\end{aligned} \tag{8.5}$$

thru the indicated order. This result is also consistent with the algebra (8.3).

Using (8.1b) and the properties of the WZW zero mode, the following results are obtained for the semiclassical WZW vertex operator:

Semiclassical unitarity of $g(\mathcal{T},\bar{z},z)$ and Lie group elements

Extreme semiclassical unitarity of the WZW vertex operator

$$g(\mathcal{T},\bar{z},z)=\exp[-2iL_g^{ab}(\bar{Q}_a^-(\bar{z})+\bar{Q}_a^+(\bar{z}))\mathcal{T}_b]G(\mathcal{T})\exp[2iL_g^{ab}(Q_a^-(z)+Q_a^+(z))\mathcal{T}_b]+\mathcal{O}(k^{-1}) \tag{8.6a}$$

$$g^\dagger(\mathcal{T},\bar{z},z)g(\mathcal{T},\bar{z},z)=\mathbb{1}+\mathcal{O}(k^{-1}) \tag{8.6b}$$

follows from the extreme semiclassical unitarity (8.2) of the WZW zero mode, or from the corresponding property of g_+ and g_-.

Moreover, we saw in (8.4) and (8.5) that the WZW zero modes satisfy the group multiplication and group integration laws in the semiclassical limit. Because the classical limit of the WZW vertex operator is the WZW zero mode

$$g(\mathcal{T},\bar{z},z)=G(\mathcal{T})+\mathcal{O}(k^{-1/2}) \tag{8.7}$$

the same multiplication and integration laws are then obtained for the classical limits of the products and averages of the WZW vertex operators,

$$\begin{aligned}&g(\mathcal{T}^1,\bar{z}_1,z_1)_{\alpha_1}{}^{\beta_1}g(\mathcal{T}^2,\bar{z}_2,z_2)_{\alpha_2}{}^{\beta_2}\\&\quad=\sum_{\mathcal{T}_k,\alpha_k,\beta_k}\bar{C}_{\alpha_1\alpha_2}{}^{\alpha_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k)C^{\beta_1\beta_2}{}_{\beta_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k)g(\mathcal{T}^k,\bar{z}_2,z_2)_{\alpha_k}{}^{\beta_k}+\mathcal{O}(k^{-1/2})\end{aligned} \tag{8.8a}$$

$$\langle 0|g(\mathcal{T}^1,\bar{z}_1,z_1)\cdots g(\mathcal{T}^n,\bar{z}_n,z_n)|0\rangle=\int \mathrm{d}\mathcal{G}\,\mathcal{G}(\mathcal{T}^1)\cdots\mathcal{G}(\mathcal{T}^n)+\mathcal{O}(k^{-1/2}) \quad . \tag{8.8b}$$

It is therefore consistent to identify the classical limit of both $g(\mathcal{T},\bar{z},z)$ and $G(\mathcal{T})$ as the classical unitary group element $\mathcal{G}(\mathcal{T})$ in irrep $\mathcal{T}$ of g,

$$g(\mathcal{T},\bar{z},z)=\mathcal{G}(\mathcal{T})+\mathcal{O}(k^{-1/2}) \tag{8.9a}$$

$$G(\mathcal{T})=\mathcal{G}(\mathcal{T})+\mathcal{O}(k^{-1/2}) \tag{8.9b}$$

$$\langle 0|(\cdots)|0\rangle=\int \mathrm{d}\mathcal{G}(\cdots)+\mathcal{O}(k^{-1/2}) \quad . \tag{8.9c}$$

To complete the classical results (8.8), the full semiclassical WZW OPE's and averages are given below.

In the same way, the classical limits of the primary states (2.5) of affine $(g\times g)$

$$\psi_\alpha{}^\beta(\mathcal{T})=g(\mathcal{T},0,0)_\alpha{}^\beta|0\rangle=\mathcal{G}(\mathcal{T})_a{}^\beta|0\rangle+\mathcal{O}(k^{-1/2}) \tag{8.10}$$

are proportional to the classical group elements. This fact was first observed in Ref.[43].

Semiclassical WZW OPE's

The full OPE of two semiclassical WZW vertex operators is

$$
\begin{aligned}
&g(\mathcal{T}^1,\bar z_1,z_1)_{\alpha_1}{}^{\beta_1} g(\mathcal{T}^2,\bar z_2,z_2)_{\alpha_2}{}^{\beta_2} \\
&= \sum_{\substack{\mathcal{T}_k,\alpha_k,\beta_k \\ \rho_1,\sigma_1}} |z_{12}|^{2[\Delta^g(\mathcal{T}^k)-\Delta^g(\mathcal{T}^1)-\Delta^g(\mathcal{T}^2)]} D^{\sigma_1\beta_2\alpha_k}_{\rho_1\alpha_2\beta_k}(\mathcal{T}^1\mathcal{T}^2\mathcal{T}^k) \\
&\times \Big\{ g(\mathcal{T}^k,\bar z_2,z_2)_{\alpha_k}{}^{\beta_k}\delta^{\rho_1}_{\alpha_1}\delta^{\beta_1}_{\sigma_1} \\
&+\sum_{r=0}^{\infty}\frac{z_{12}^{r+1}}{(r+1)!}2L_g^{ab} :g(\mathcal{T}^k,\bar z_2,z_2)_{\alpha_k}{}^{\beta_k}\partial_2^r J_b(z_2): \delta^{\rho_1}_{\alpha_1}(\mathcal{T}^1_a)_{\sigma_1}{}^{\beta_1} \\
&-\sum_{r=0}^{\infty}\frac{\bar z_{12}^{r+1}}{(r+1)!}2L_g^{ab} :g(\mathcal{T}^k,\bar z_2,z_2)_{\alpha_k}{}^{\beta_k}\bar\partial_2^r \bar J_b(\bar z_2): (\mathcal{T}^1_a)_{\alpha_1}{}^{\rho_1}\delta^{\beta_1}_{\sigma_1} \\
&+\sum_{r,s=0}^{\infty}\frac{z_{12}^{r+s+2}}{(r+s+2)!}4L_g^{ab}L_g^{cd} :g(\mathcal{T}^k,\bar z_2,z_2)_{\alpha_k}{}^{\beta_k}\partial_2^r[\partial_2^s J_b(z_2)J_c(z_2)]: \delta^{\rho_1}_{\alpha_1}(\mathcal{T}^1_d\mathcal{T}^1_a)_{\sigma_1}{}^{\beta_1} \\
&+\sum_{r,s=0}^{\infty}\frac{\bar z_{12}^{r+s+2}}{(r+s+2)!}4L_g^{ab}L_g^{cd} :g(\mathcal{T}^k,\bar z_2,z_2)_{\alpha_k}{}^{\beta_k}\bar\partial_2^r[\bar\partial_2^s \bar J_b(\bar z_2)\bar J_c(\bar z_2)]: (\mathcal{T}^1_a\mathcal{T}^1_d)_{\alpha_1}{}^{\rho_1}\delta^{\beta_1}_{\sigma_1} \\
&-\sum_{r,s=0}^{\infty}\frac{z_{12}^{r+1}}{(r+1)!}\frac{\bar z_{12}^{s+1}}{(s+1)!}4L_g^{ab}L_g^{cd} :g(\mathcal{T}^k,\bar z_2,z_2)_{\alpha_k}{}^{\beta_k}\partial_2^r J_b(z_2)\bar\partial_2^s\bar J_c(\bar z_2): (\mathcal{T}^1_d)_{\alpha_1}{}^{\rho_1}(\mathcal{T}^1_a)_{\sigma_1}{}^{\beta_1}\Big\} \\
&\qquad + \mathcal{O}(k^{-3/2})
\end{aligned}
\tag{8.11}
$$

where D in (8.4) is quadratic in the Clebsch-Gordan coefficients C. It is observed that these OPE's have trivial monodromy, as they should, with leading semiclassical terms which are the WZW primary fields themselves.

Semiclassical WZW correlators

The semiclassical WZW correlators are

$$
A_g(\mathcal{T},\bar z,z) = A_g^-(\mathcal{T},\bar z)A_g^+(\mathcal{T},z) = \langle 0|g(\mathcal{T}^1,\bar z_1,z_1)\cdots g(\mathcal{T}^n,\bar z_n,z_n)|0\rangle \tag{8.12a}
$$

$$
= \left[1+2L_g^{ab}\sum_{i<j}^n \mathcal{T}^i_a\mathcal{T}^j_b \ln \bar z_{ij}\right]\langle 0|G(\mathcal{T}^1)\cdots G(\mathcal{T}^n)|0\rangle\left[1+2L_g^{ab}\sum_{i<j}^n \mathcal{T}^i_a\mathcal{T}^j_b \ln z_{ij}\right] + \mathcal{O}(k^{-3/2}) \tag{8.12b}
$$

$$
= \left[1+2L_g^{ab}\sum_{i<j}^n \mathcal{T}^i_a\mathcal{T}^j_b \ln \bar z_{ij}\right] I_g^n \left[1+2L_g^{ab}\sum_{i<j}^n \mathcal{T}^i_a\mathcal{T}^j_b \ln z_{ij}\right] + \mathcal{O}(k^{-3/2}) \tag{8.12c}
$$

$$
= \left[1+4L_g^{ab}\sum_{i<j}^n \mathcal{T}^i_a\mathcal{T}^j_b \ln |z_{ij}|\right] I_g^n + \mathcal{O}(k^{-3/2}) \tag{8.12d}
$$

wher I_g^n is the Haar integral in (2.17). The result (8.12), which is the central check on the results of this paper, is the known [42] form of the semiclassical WZW correlators. The simplicity of the result is due to the fact that, once again, the normal-ordered terms in (8.1b) fail to contribute to the correlators at this order of the semiclassical expansion. It is clear that these correlators have trivial monodromy when one z goes around another, and, moreover, the chiral and antichiral intrinsic monodromies (4.30) have cancelled, so that the intrinsic monodromies of A_g are trivial.

9 Conclusions

Supplementing the discussion of Moore and Reshetikhin [25] and others [26-41] we have given a new semiclassical nonabelian vertex operator construction of the chiral and antichiral primary fields (the chiral and antichiral vertex operators) associated to WZW theory, and the nonchiral primary fields of WZW theory itself.

The new nonabelian vertex operators were obtained as the explicit semiclassical solution of known [26,25] operator differential equations for the chiral and antichiral primary fields, and they are the natural nonabelian generalization of the familiar abelian vertex operators [23]: The new vertex operators involve only the representation matrices $\mathcal{T}$ of Lie g, the currents $J, \bar{J}$ of affine $(g \times g)$ and the chiral and antichiral zero modes $G_{\pm}(\mathcal{T})$, and they reduce to the familiar abelian vertex operators in the limit of abelian algebras. So far as we have carried out the semiclassical expansion, it was seen that the zero modes carry the full action of the quantum group, and moreover, we were able to identify the classical limit of the nonchiral WZW zero mode $G(\mathcal{T}) = G_{-}(\mathcal{T})G_{+}(\mathcal{T})$ as the classical group element in irrep $\mathcal{T}$ of g.

Combining our results with those of Ref.[25], we computed the semiclassical OPE's among the chiral and antichiral vertex operators, and among the nonchiral WZW vertex operators themselves. Moreover, it was verified that the new vertex operators reproduce the known [42] form of the semiclassical affine-Sugawara conformal blocks and WZW correlators, and connections with semiclassical crossing matrices [42] and braid relations [30,40] were also discussed.

We finally note that semiclassical blocks and correlators are also known [42] for the coset constructions and a class of processes in irrational conformal field theory. Consequently, the present work should be considered as a first step toward finding the semiclassical nonabelian vertex operators and OPE's of these more general theories.

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